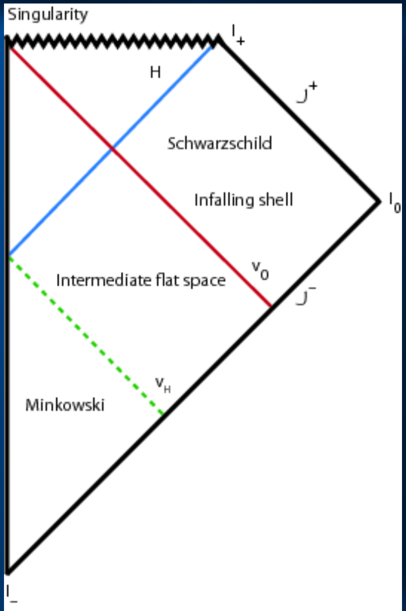
GARRA Group

Black hole astrophysics



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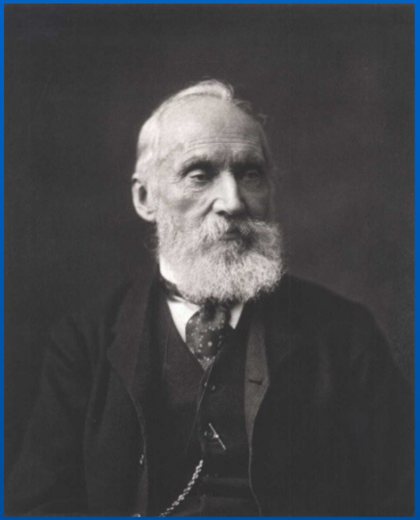
Curso 2022

What is a star?

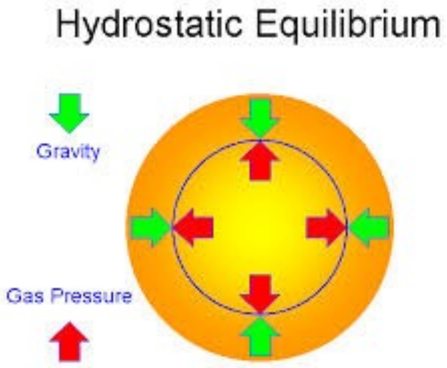


The idea that stars are self-gravitating gaseous bodies was introduced in theXIX Century by Lane, Kelvin and Helmholtz. They suggested that stars should be understood in terms of the equation of hydrostatic equilibrium:

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2},$$

where the pressure P is given by

$$P = \frac{\rho k T}{\mu m_p}.$$



What is a star?

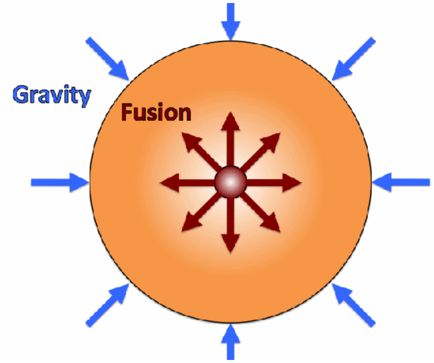
Eddington proposed (1926):

1. Thermonuclear reactions are the source of energy in the stars
2. The outward pressure of radiation should be taken into account in the equation for equilibrium.

$$\frac{d}{dr}\left[\frac{\rho kT}{\mu m_p} + \frac{1}{3}aT^4\right] = -\frac{GM(r)\rho(r)}{r^2},$$

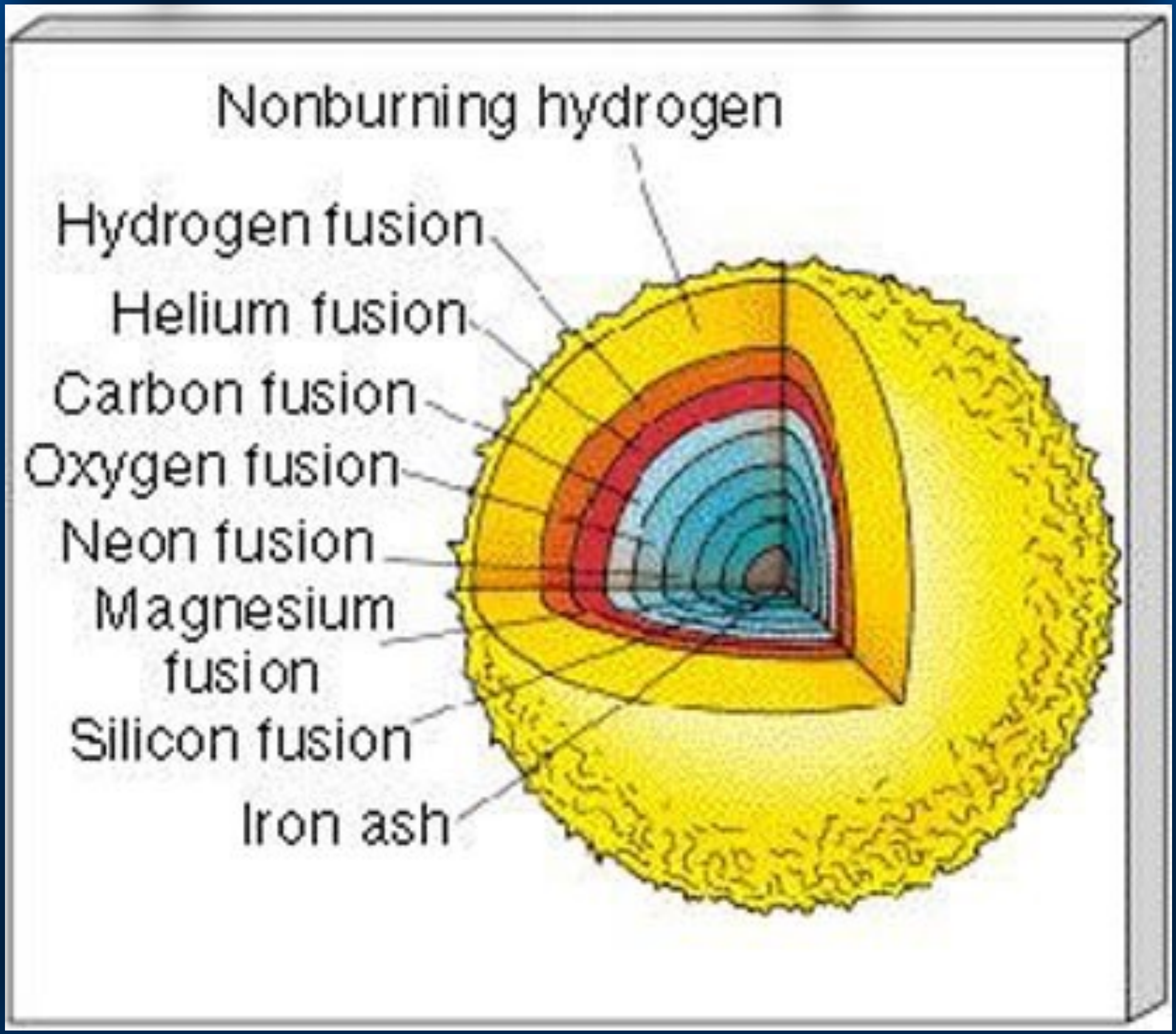
$$\frac{dP_{\rm rad}(r)}{dr} = -\left(\frac{L(r)}{4\pi r^2 c}\right)\frac{1}{l},$$

$$\frac{dL(r)}{dr} = 4\pi r^2 \epsilon\rho,$$

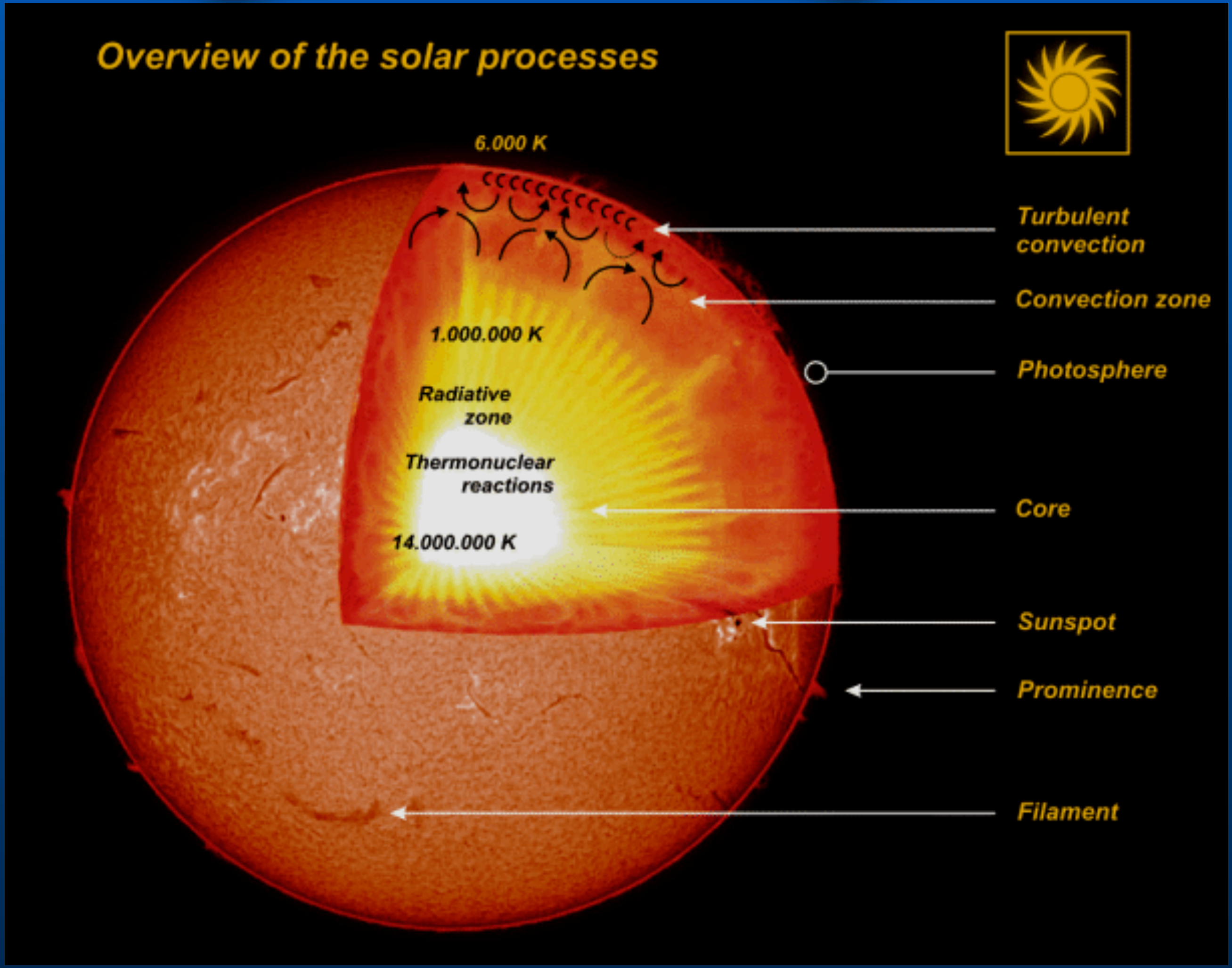


where l is the mean free path of the photons, L the luminosity, and ϵ the energy generated per gram of material per unit time.

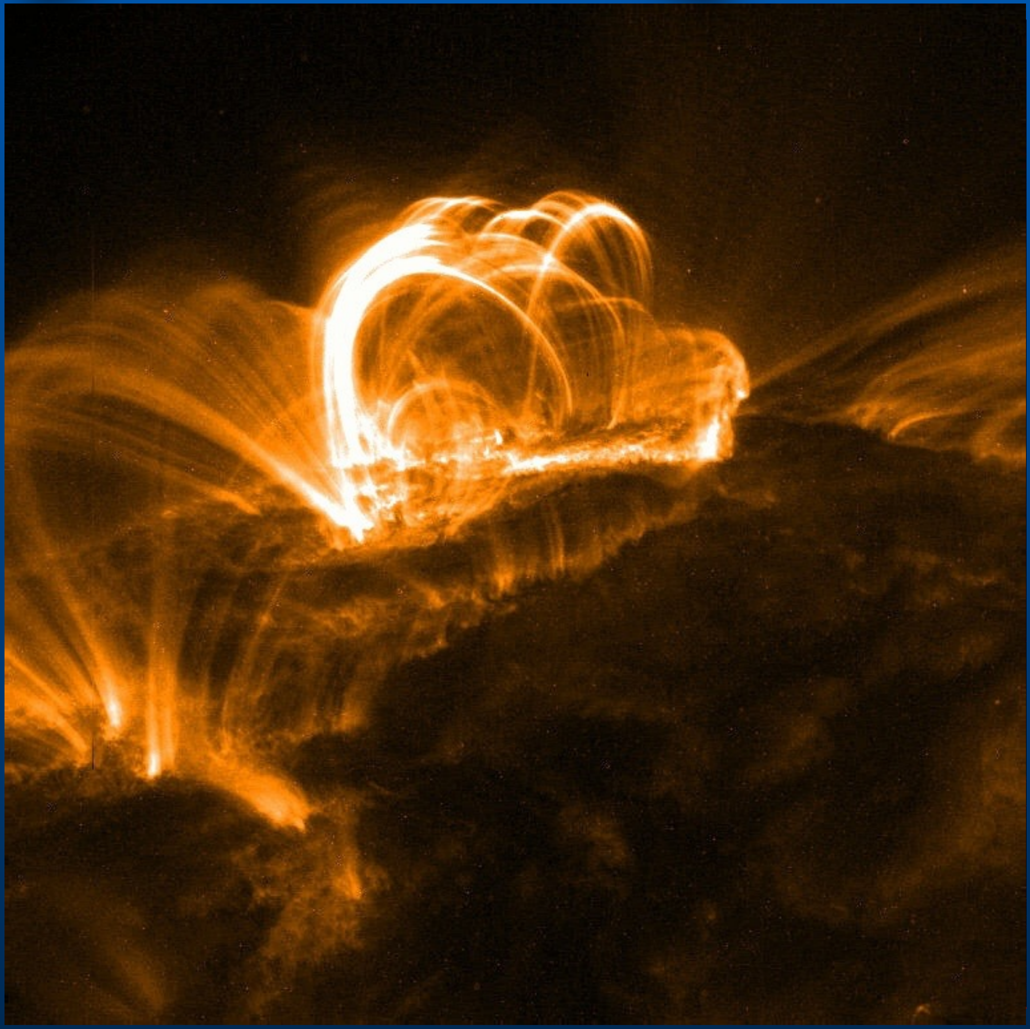
What is a star?



What is a star?



What is a star?



What is a star?

Once the nuclear power of the star is exhausted, the contribution from the radiation pressure decreases dramatically when the temperature diminishes. The star then contracts until a new pressure helps to balance gravity attraction: the degeneracy pressure of the electrons. The equation of state for a degenerate gas of electrons is:

$$P_{\rm rel} = K\rho^{4/3}.$$

Then,

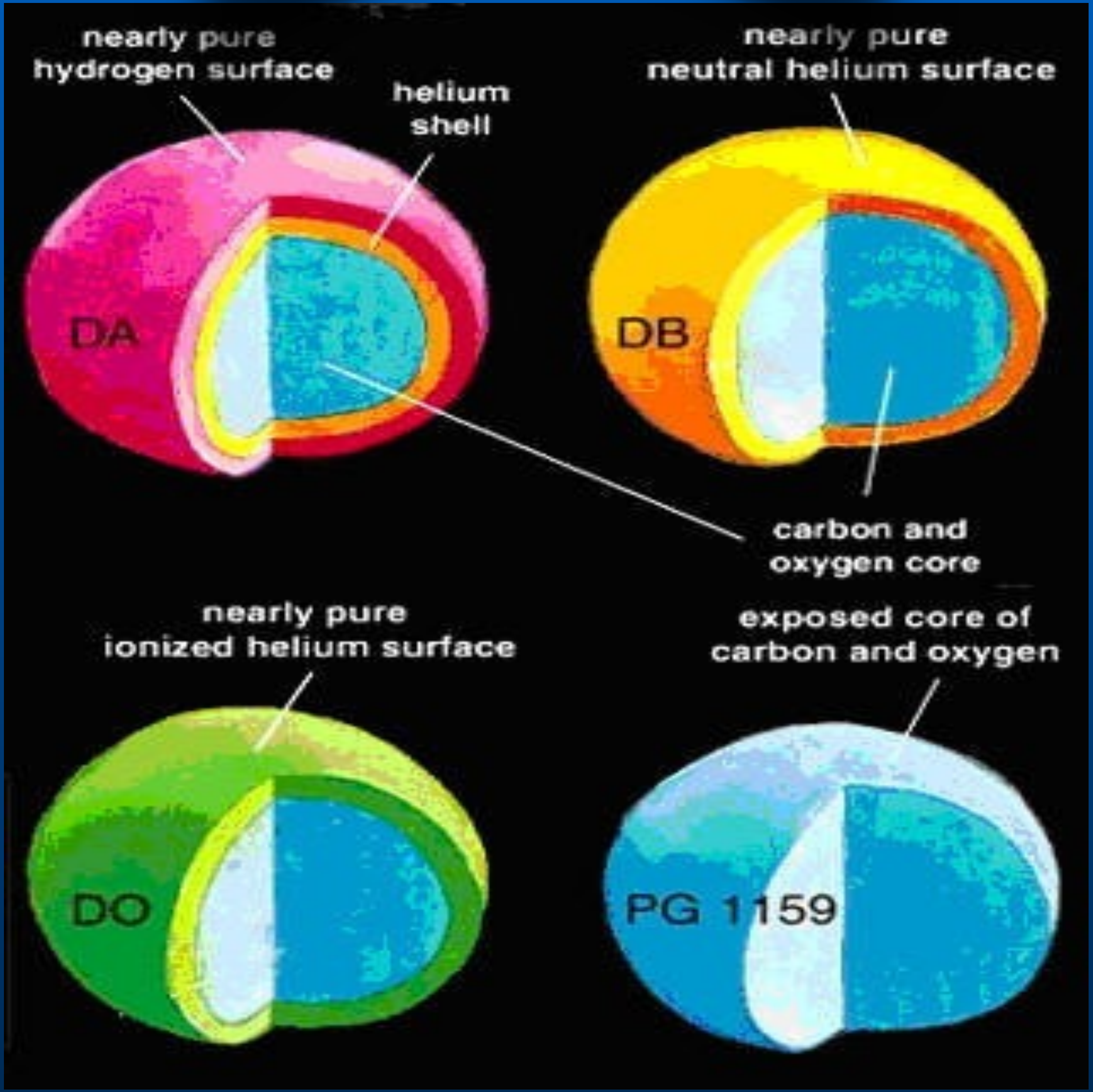
$$\frac{M^{4/3}}{r^5} \propto \frac{GM^2}{r^5}.$$

Since the radius cancels out, this relations can be satisfied by a unique mass:

$$M = 0.197\left[\left(\frac{hc}{G}\right)^3 \frac{1}{m_p^2}\right] \frac{1}{\mu_e^2} = 1.4 \ \ M_\odot,$$

where μ_e is the mean molecular weight of the electrons. The result implies that a completely degenerated star have this and only this mass. This limit was found by Chandrasekhar (1931) and is known as the *Chandrasekhar limit*.

What is a star?



What is a star?

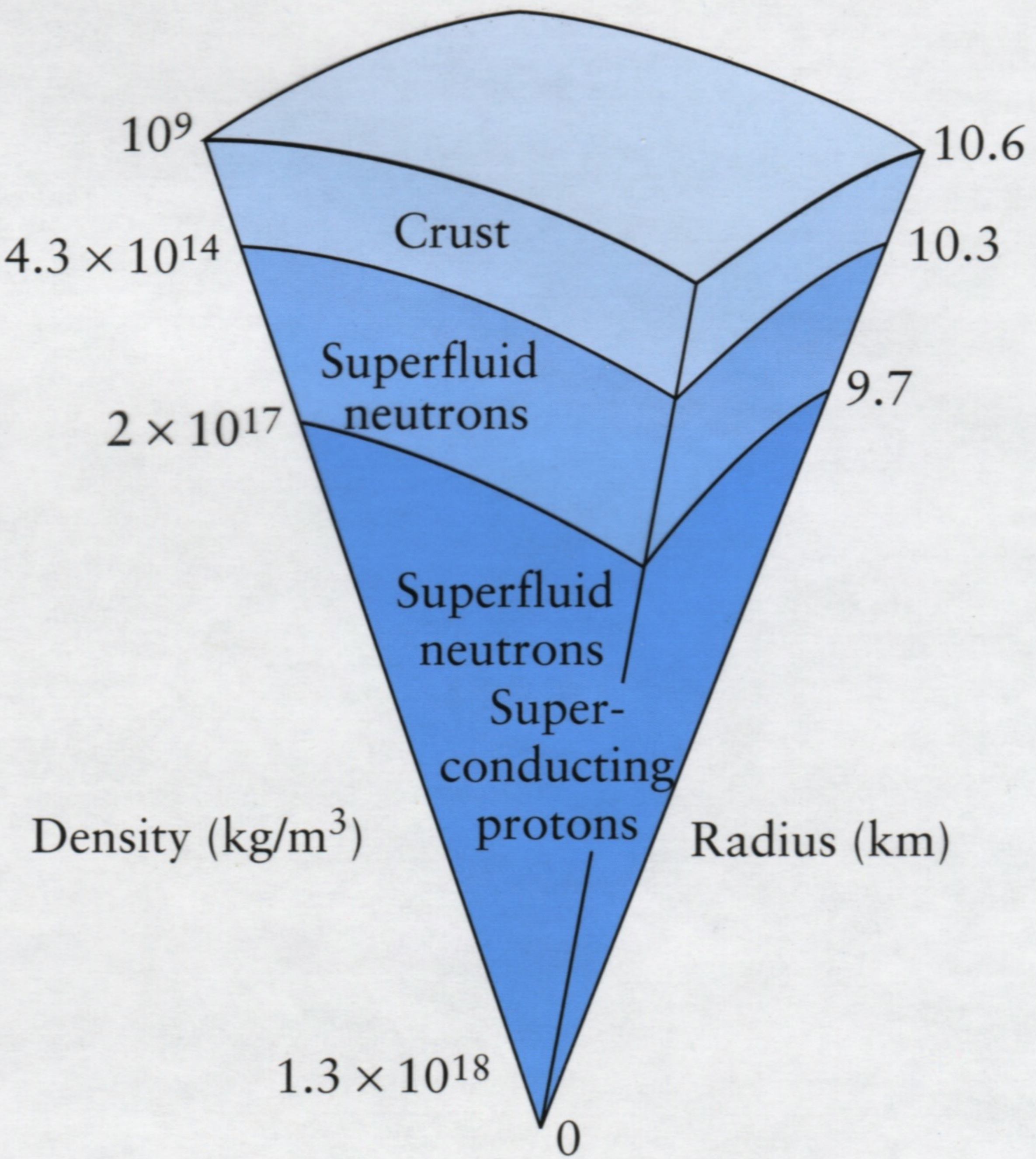
Fritz Zwicky
1898-1974

JANUARY 15, 1934 PHYSICAL REVIEW VOLU

Proceedings of the American Physical Society

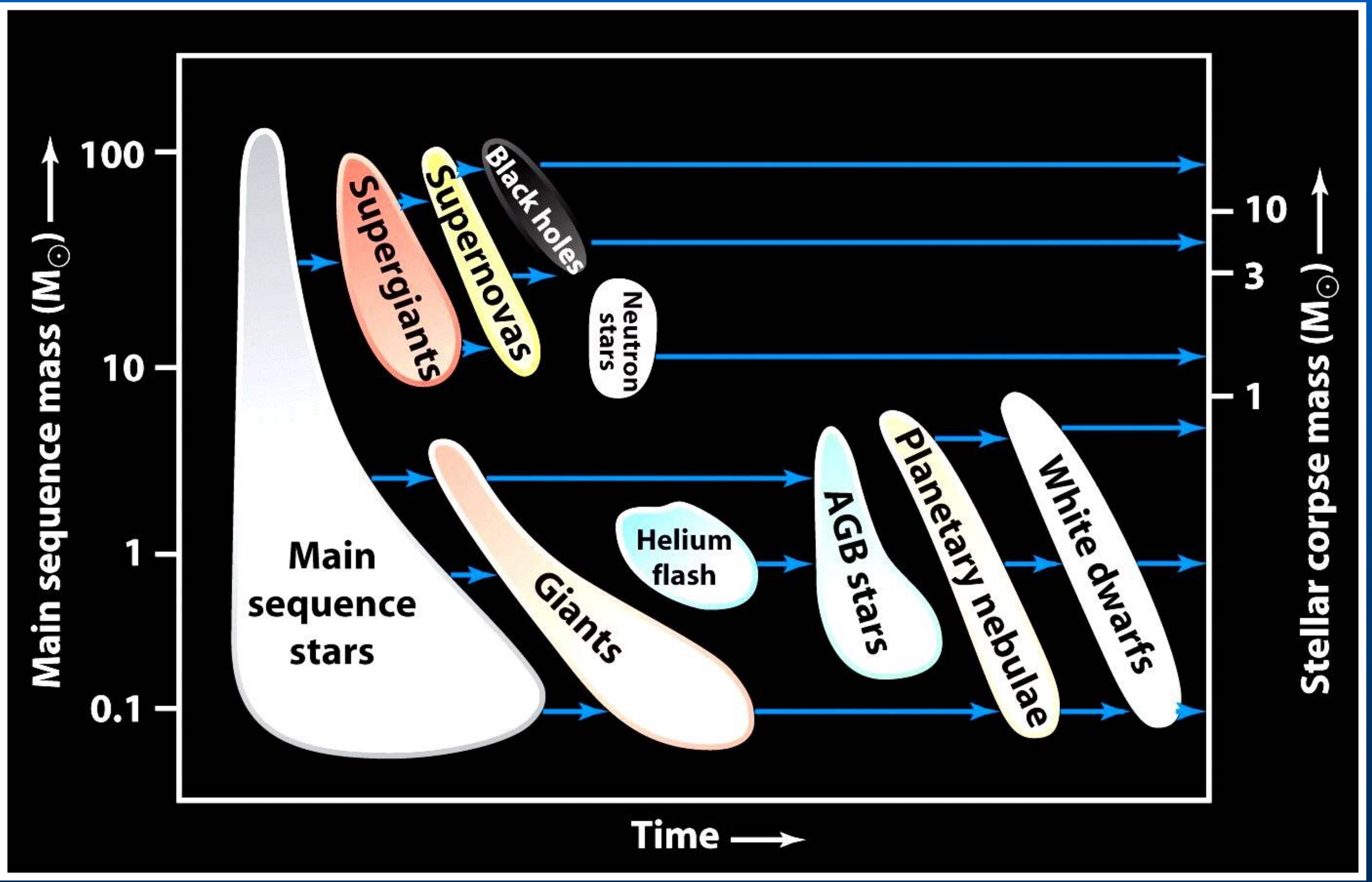
MINUTES OF THE STANFORD MEETING, DECEMBER 15–16, 1933

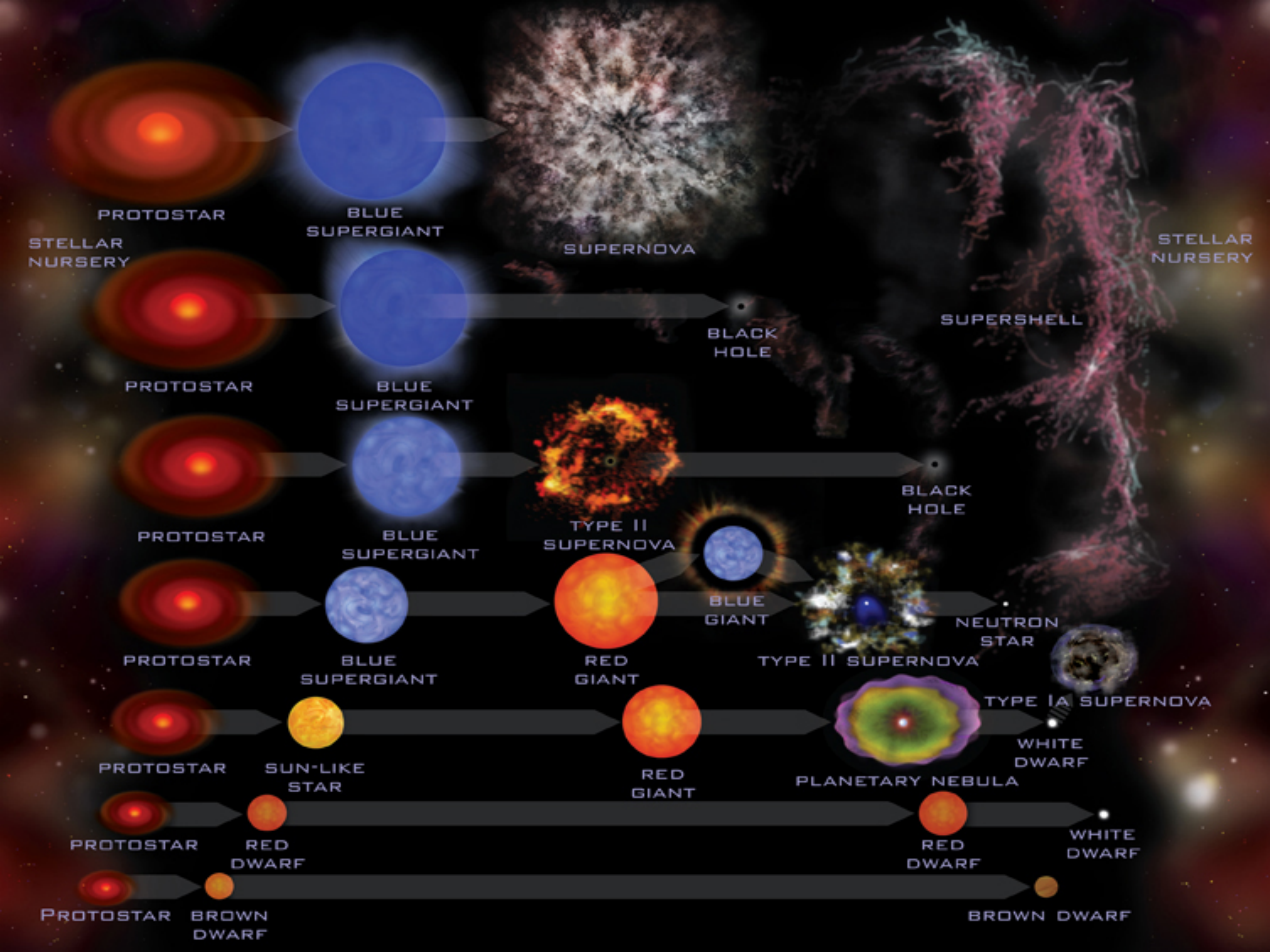
38. Supernovae and Cosmic Rays. W. BAADE, *Mt. Wilson Observatory*, AND F. ZWICKY, *California Institute of Technology*.—Supernovae flare up in every stellar system (nebula) once in several centuries. The lifetime of a supernova is about twenty days and its absolute brightness at maximum may be as high as $M_{vis} = -14^M$. The visible radiation L_v of a supernova is about 10^8 times the radiation of our sun, that is, $L_v = 3.78 \times 10^{41}$ ergs/sec. Calculations indicate that the total radiation, visible and invisible, is of the order $L_\tau = 10^7 L_v = 3.78 \times 10^{48}$ ergs/sec. The supernova therefore emits during its life a total energy $E_\tau \geq 10^5 L_\tau = 3.78 \times 10^{53}$ ergs. If supernovae initially are quite ordinary stars of mass $M < 10^{34}$ g. E_τ… same order as M itself. In the *supernova* pro… *bulk is annihilated*. In addition the hypothe… itself that *cosmic rays are produced by supernov*… that in every nebula one supernova occurs eve… years, the intensity of the cosmic rays to be… the earth should be of the order $\sigma = 2 \times 10^{-3}$… The observational values are about $\sigma = 3 \times$… sec. (Millikan, Regener). With all reserve we… view that supernovae represent the trans… ordinary stars into *neutron stars*, which in thei… consist of extremely closely packed neutrons



10^9
10.6
Crust
4.3×10^{14}
10.3
Superfluid neutrons
9.7
2×10^{17}
Superfluid neutrons
Super-conducting protons
Density (kg/m^3)
Radius (km)
1.3×10^{18}
0

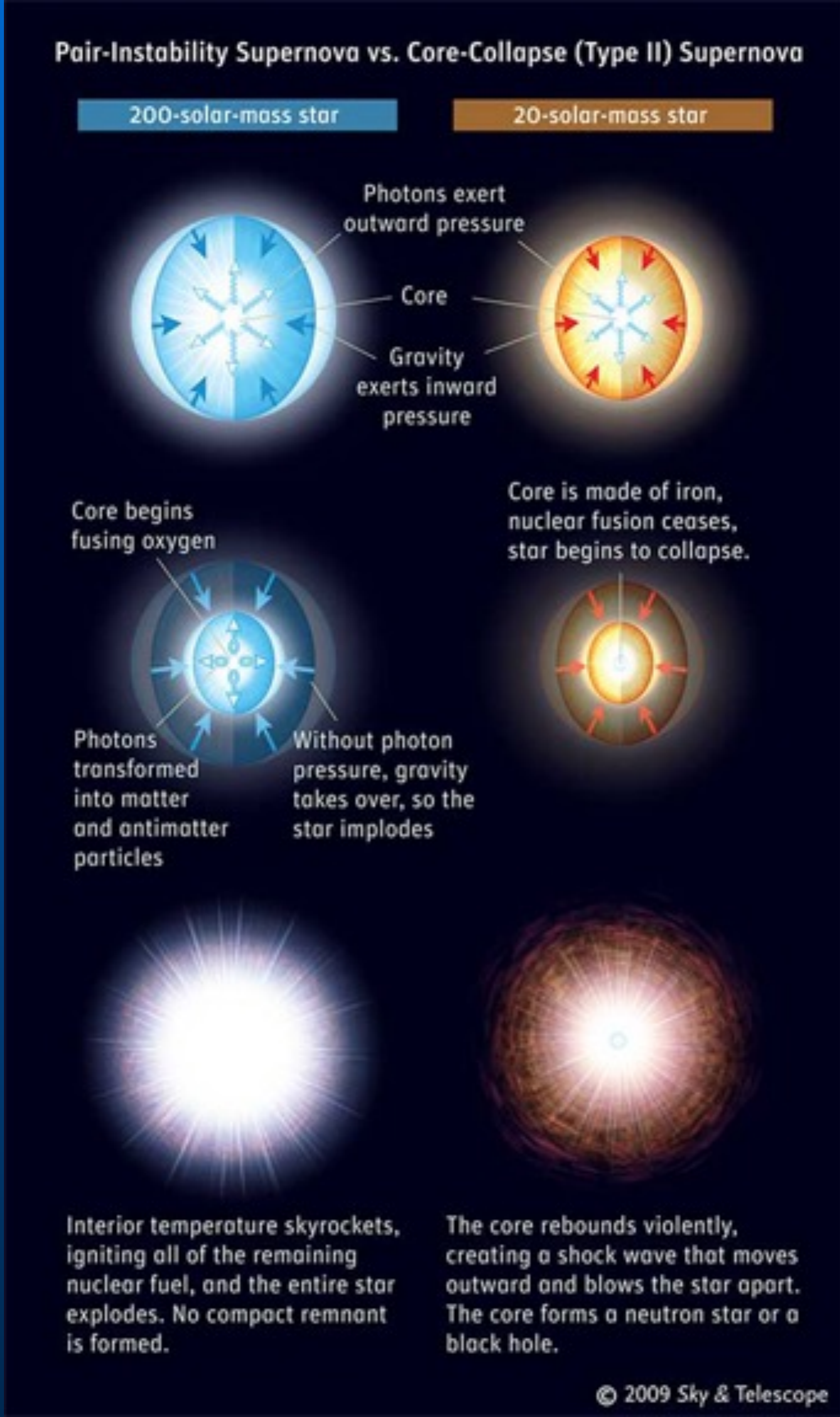
The end of stars

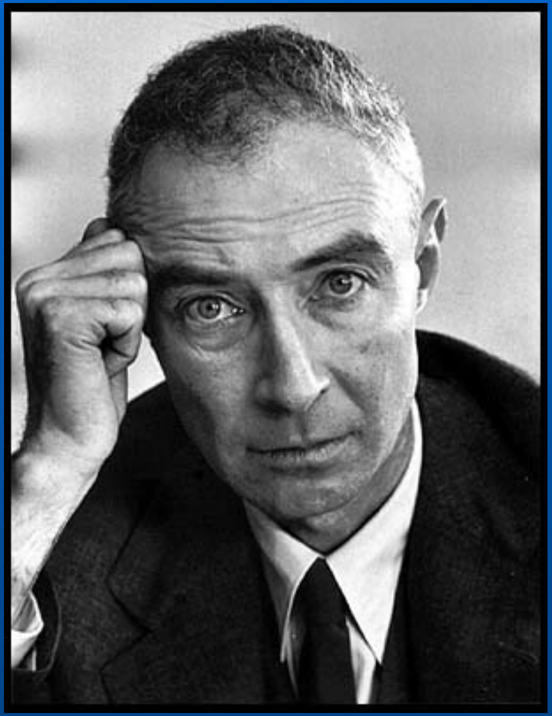



PROTOSTAR
BLUE SUPERGIANT
SUPERNOVA
STELLAR NURSERY
STELLAR NURSERY
PROTOSTAR
BLUE SUPERGIANT
BLACK HOLE
SUPERSHELL
PROTOSTAR
BLUE SUPERGIANT
TYPE II SUPERNOVA
BLACK HOLE
PROTOSTAR
BLUE SUPERGIANT
RED GIANT
BLUE GIANT
TYPE II SUPERNOVA
NEUTRON STAR
TYPE IA SUPERNOVA
PROTOSTAR
SUN-LIKE STAR
RED GIANT
PLANETARY NEBULA
WHITE DWARF
PROTOSTAR
RED DWARF
RED DWARF
WHITE DWARF
PROTOSTAR
BROWN DWARF
BROWN DWARF

A **pair-instability supernova** occurs when pair production, the production of free electrons and positrons in the collision between atomic nuclei and energetic gamma rays, reduces thermal pressure inside a supermassive star's core. This pressure drop leads to a partial collapse, then greatly accelerated burning in a runaway thermonuclear explosion which blows the star completely apart without leaving a black hole remnant behind.

Pair-instability supernovae can only happen in stars with a **mass range from around 130 to 250 solar masses** and low to moderate metallicity (low abundance of elements other than hydrogen and helium, a situation common in Population III stars).

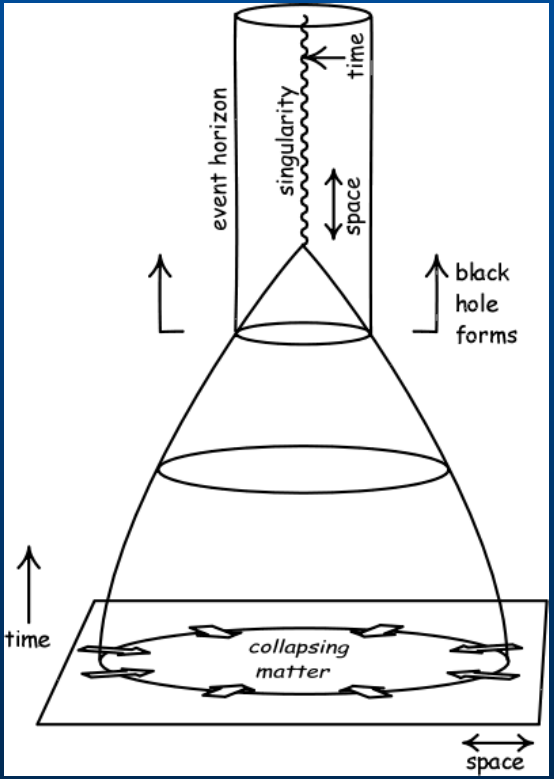




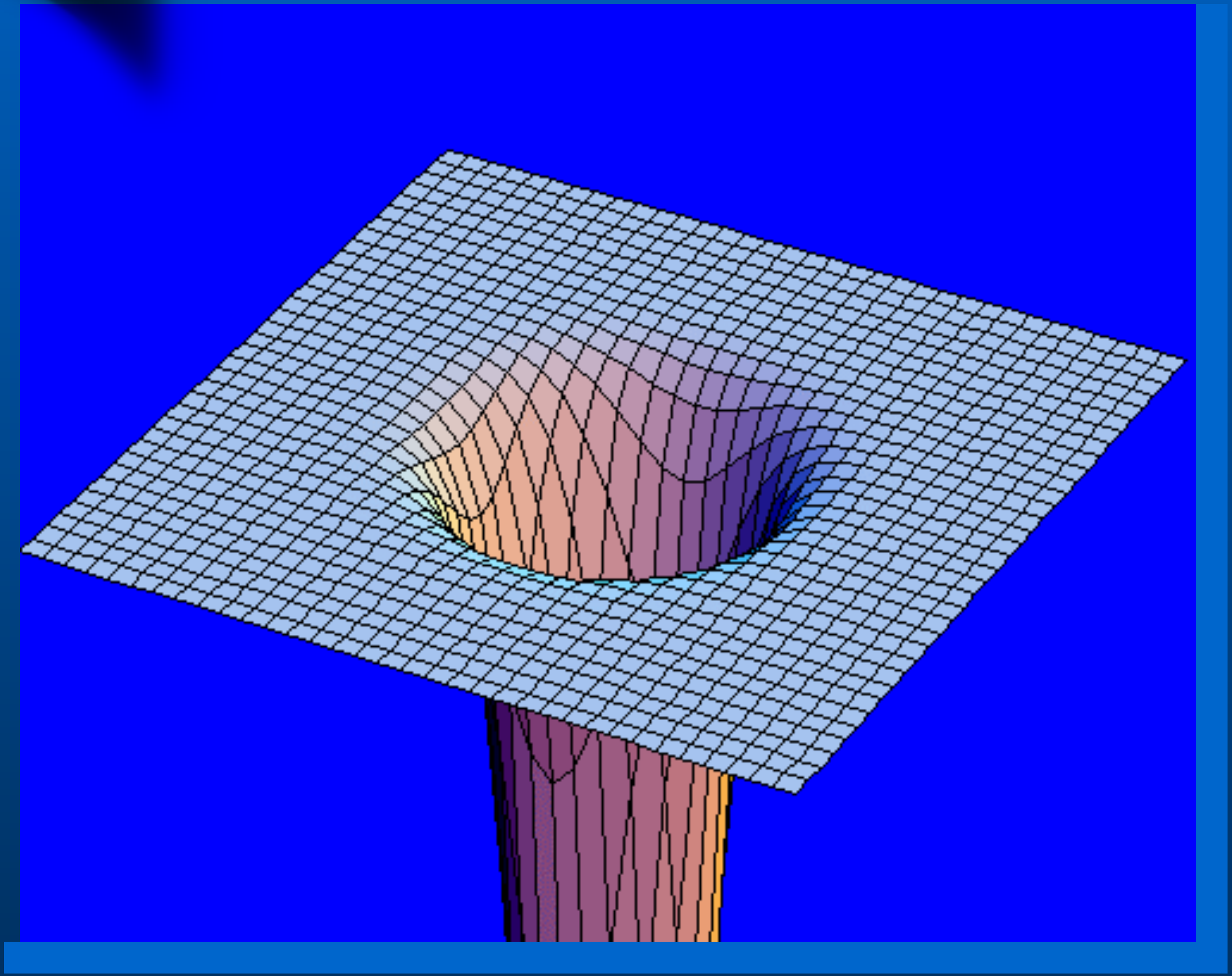
Oppenheimer & Snyder (1939):
non-stopping collapse.

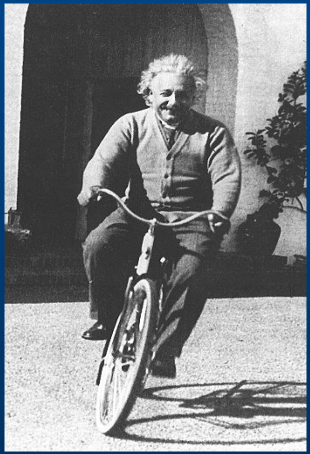
Collapse to what?

The answer is in General Relativity.



Black holes



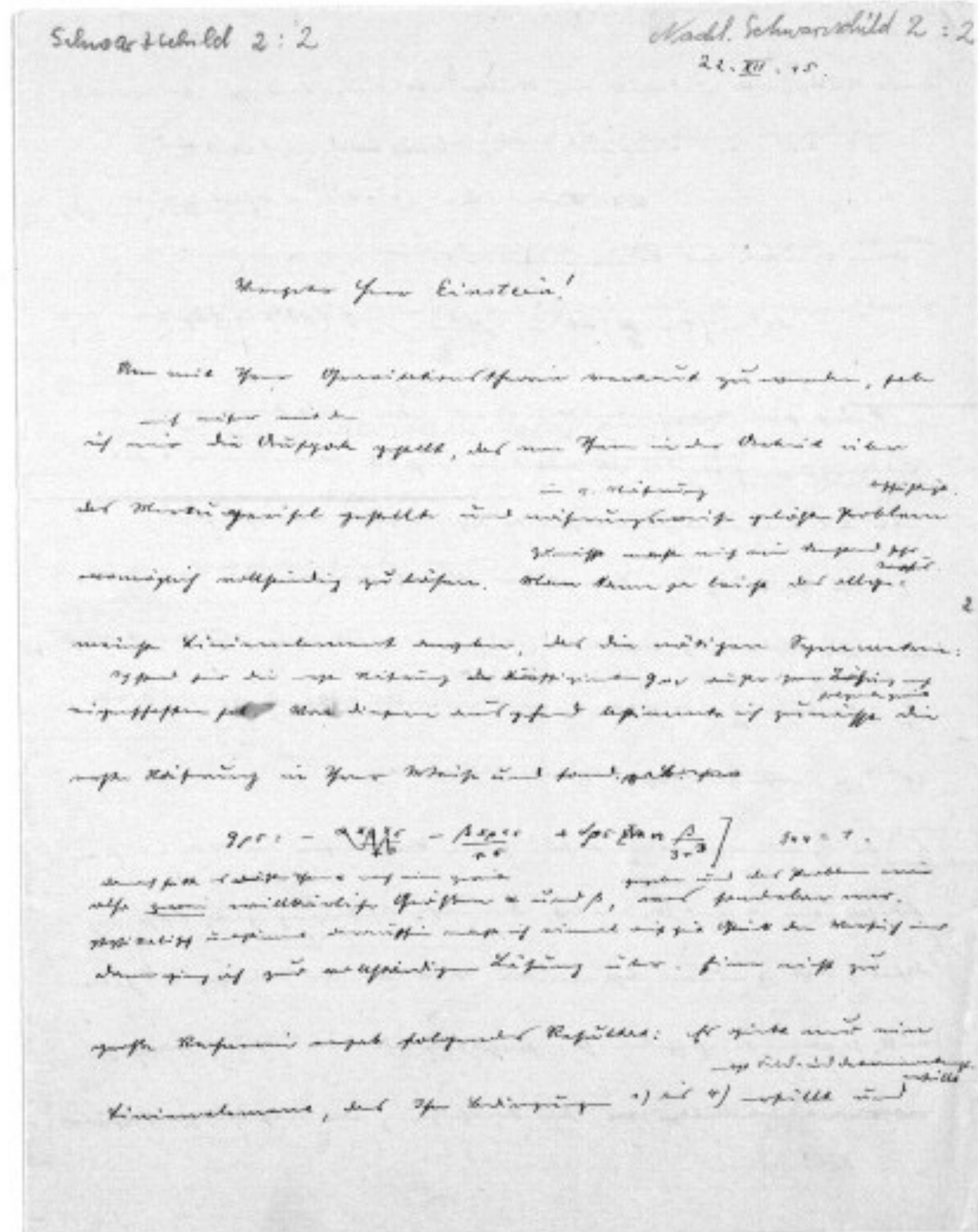


Albert Einstein

John A. Wheeler

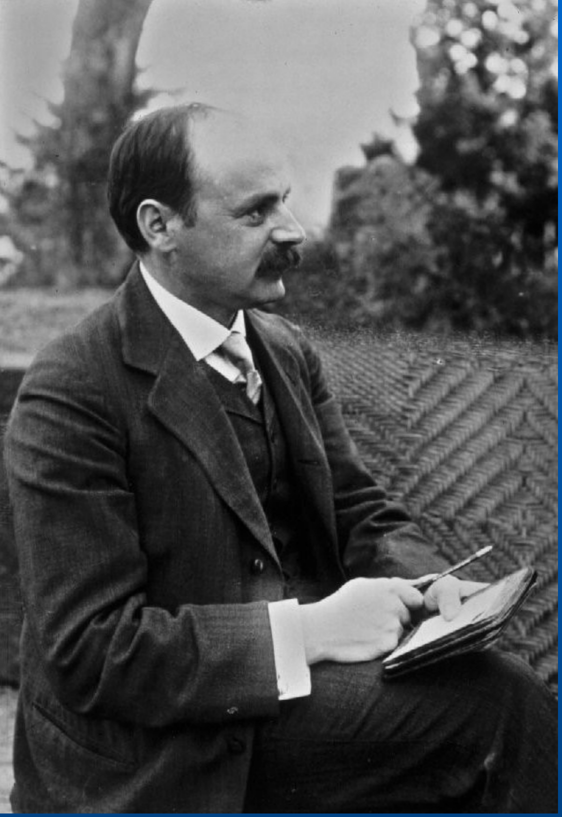
Letter
from Karl Schwarzschild to Einstein, 22 December 1915

Schwarzschild is one of the few astronomers who are interested in Einstein's General Theory of Relativity. In December 1915, he is based at the Russian front line. However he finds the opportunity to deduce the first exact solution of Einstein's field equations. “As you see, the war is friendly to me”, he writes.


Schwarzschild 2:2
Nachl. Schwarzschild 2:2
22.XII.15.

<u>On the death of Karl Schwarzschild</u>

When war was declared in 1914, Schwarzschild volunteered for the German army and manned weather stations and calculated missile trajectories in France, Belgium, and Russia. It was in Russia that he discovered and published his well-known results in relativity as well as a derivation of the Stark effect using the 'old' quantum mechanics. It was also in Russia that he began to struggle with *pemphigus*, an autoimmune disease where the body starts attacking its own cells. He was sent home, where he died on May 11, 1916 at the age of 42.



Vacuum spherically symmetric solution to Einstein equations.

$$R_{\mu\nu} = 0.$$

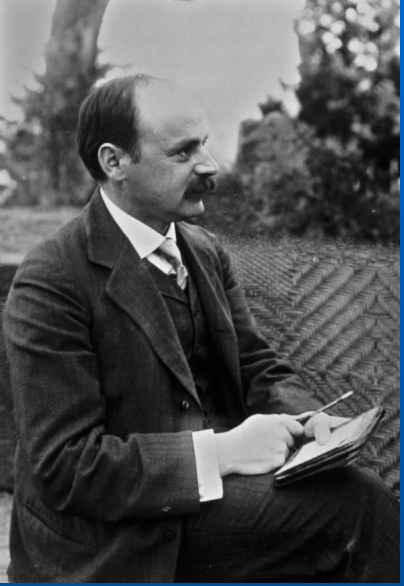
Karl Schwarzschild (1916)

The first exact solution of Einstein field equations was found by Karl Schwarzschild in 1916. This solution describes the geometry of space-time outside a spherically symmetric matter distribution.

The most general spherically symmetric metric is:

$$ds^2 = \alpha(r,\ t)dt^2 - \beta(r,\ t)dr^2 - \gamma(r,\ t)d\Omega^2 - \delta(r,\ t)drdt,$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. We are using spherical polar coordinates. The metric (133) is invariant under rotations (isotropic).



$$x'^{\mu} = f^{\mu}(x)$$

The invariance group of general relativity is formed by the group of general transformations of coordinates.This yields 4 degrees of freedom, two of which have been used when adopting spherical coordinates. With the two available degrees of freedom we can freely choose two metric coefficients, whereas the other two are determined by Einstein's equations.

Schwarzschild solution.

- *Standard gauge.*

$$ds^2 = c^2 A(r,\ t)dt^2 - B(r,\ t)dr^2 - r^2 d\Omega^2.$$

- *Synchronous gauge.*

$$ds^2 = c^2 dt^2 - F^2(r,\ t)dr^2 - R^2(r,\ t)d\Omega^2.$$

- *Isotropic gauge.*

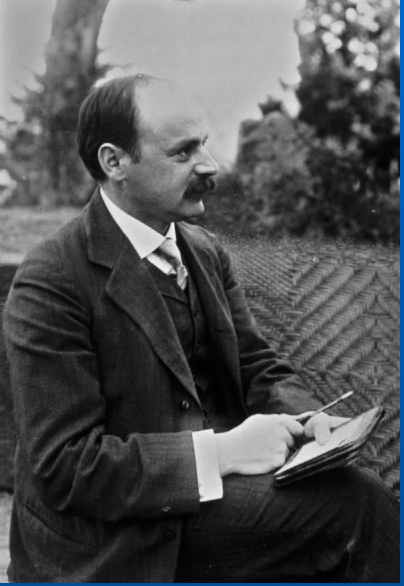
$$ds^2 = c^2 H^2(r,\ t)dt^2 - K^2(r,\ t)\left[dr^2 + r^2(r,\ t)d\Omega^2\right].$$

- *Co-moving gauge.*

$$ds^2 = c^2 W^2(r,\ t)dt^2 - U(r,\ t)dr^2 - V(r,\ t)d\Omega^2.$$

Static

$$ds^2 = c^2 A(r)dt^2 - B(r)dr^2 - r^2 d\Omega^2.$$



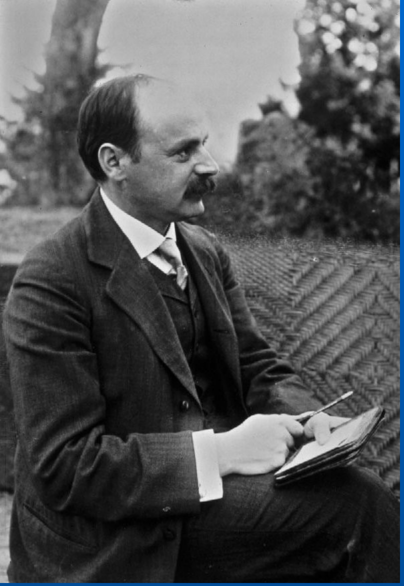
Schwarzschild solution.

$$ds^2 = c^2 A(r) dt^2 - B(r) dr^2 - r^2 d\Omega^2.$$

$$R_{\mu\nu} = 0.$$

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} (\partial_\nu g_{\rho\mu} + \partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu}),$$

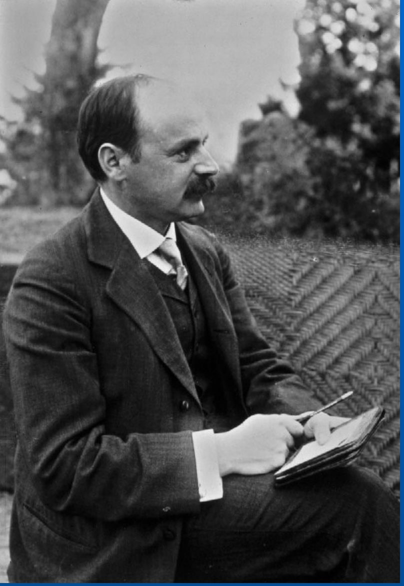
$$R_{\mu\nu} = \partial_\nu \Gamma^{\sigma}_{\mu\sigma} - \partial_\sigma \Gamma^{\sigma}_{\mu\nu} + \Gamma^{\rho}_{\mu\sigma} \Gamma^{\sigma}_{\rho\nu} - \Gamma^{\rho}_{\mu\nu} \Gamma^{\sigma}_{\rho\sigma} = 0.$$



Schwarzschild solution.

The metric coefficients are:

$$\begin{aligned}
g_{00} &= c^2 A(r), \\
g_{11} &= -B(r), \\
g_{22} &= -r^2, \\
g_{33} &= -r^2 \sin^2\theta, \\
g^{00} &= 1/A(r), \\
g^{11} &= -1/B(r), \\
g^{22} &= -1/r^2, \\
g^{33} &= -1/r^2 \sin^2\theta.
\end{aligned}$$

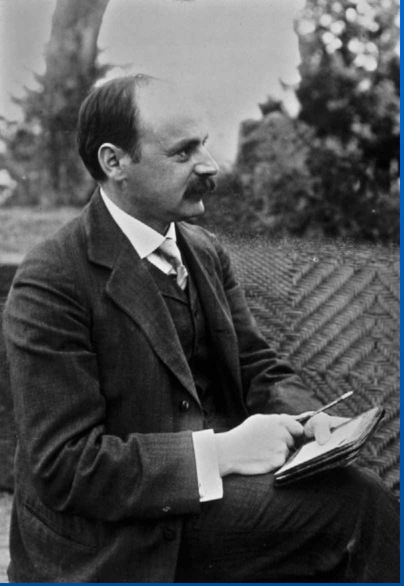


Schwarzschild solution.

Then, only nine of the 40 independent connection coefficients are different from zero. They are:

$$
\begin{aligned}
\Gamma^1_{01} &= A'/(2A), \\
\Gamma^1_{22} &= -r/B, \\
\Gamma^2_{33} &= -\sin\theta\cos\theta, \\
\Gamma^1_{00} &= A'/(2B), \\
\Gamma^1_{33} &= -(r\sin^2/B), \\
\Gamma^3_{13} &= 1/r, \\
\Gamma^1_{11} &= B'/(2B), \\
\Gamma^2_{12} &= 1/r, \\
\Gamma^3_{23} &= \cot\theta, .
\end{aligned}
$$

Schwarzschild solution.



Replacing in the expression for $R_{\mu\nu}$:

$$
\begin{aligned}
R_{00} &= -\frac{A''}{2B} + \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rB}, \\
R_{11} &= \frac{A''}{2A} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rB}, \\
R_{22} &= \frac{1}{B} - 1 + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right), \\
R_{33} &= R_{22}\sin^2\theta.
\end{aligned}
$$

The Einstein field equations for the region of empty space then become:

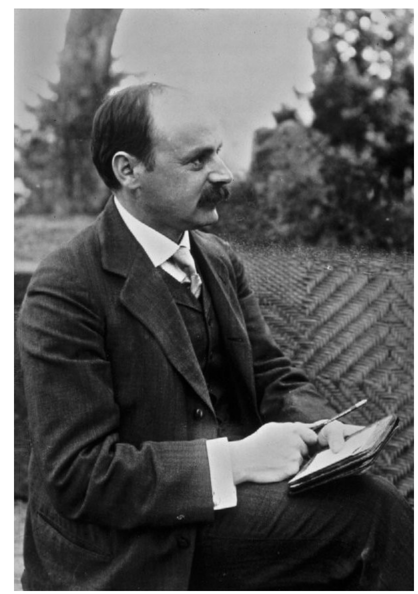
$$R_{00} = R_{11} = R_{22} = 0$$

(the fourth equation has no additional information). Multiplying the first equation by B/A and adding the result to the second equation, we get:

$$A'B + AB' = 0,$$

from which $AB =$ constant. We can write then $B = \alpha A^{-1}$. Going to the third equation and replacing B we obtain: $A + rA' = \alpha$, or:

$$\frac{d(rA)}{dr} = \alpha.$$

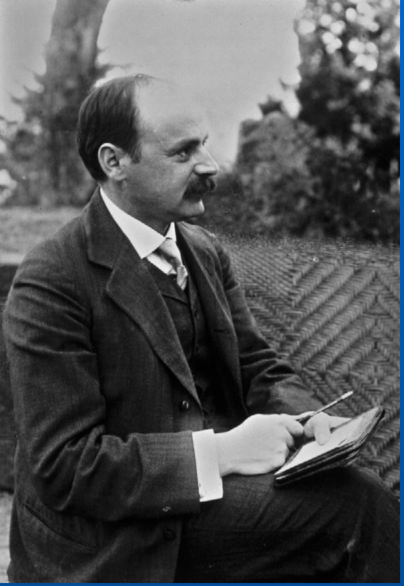


The solution of this equation is:

$$A(r) = \alpha\left(1 + \frac{k}{r}\right),$$

with k another integration constant. For B we get:

$$B = \left(1 + \frac{k}{r}\right)^{-1}.$$



Schwarzschild solution.

If now we consider the Newtonian limit:

$$\frac{A(r)}{c^2} = 1 + \frac{2\Phi}{c^2},$$

with Φ the Newtonian gravitational potential: $\Phi = -GM/r$, we can conclude that

$$k = -\frac{2GM}{c^2}$$

and

$$\alpha = c^2.$$

Spherically symmetric black holes

The solution of Einstein's equations for the vacuum region exterior to a spherical object of mass *M* is:

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right)c^2dt^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

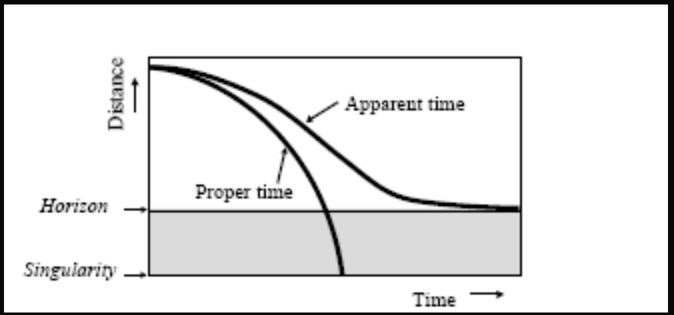
$$r_{\mathrm{Schw}} = \frac{2GM}{c^2}.$$

$$d\tau = \left(1 - \frac{2GM}{rc^2}\right)^{1/2}dt,$$

$$dt = \left(1 - \frac{2GM}{rc^2}\right)^{-1/2}d\tau.$$

$$z = \frac{\lambda_\infty - \lambda}{\lambda},$$

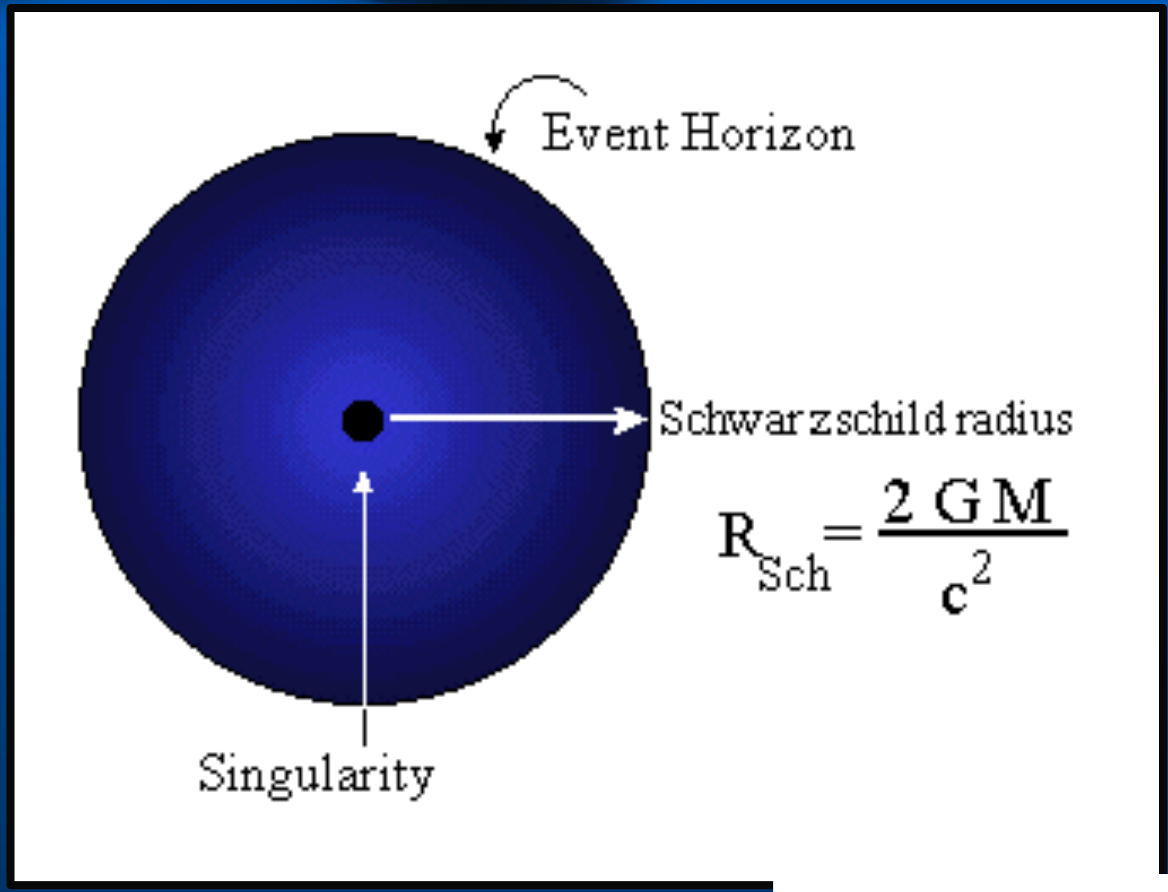
Strange things occur at the Schwarzschild radius.



$$\lambda_\infty = \left(1 - \frac{2GM}{rc^2}\right)^{-1/2}\lambda.$$

$$1 + z = \left(1 - \frac{2GM}{rc^2}\right)^{-1/2},$$

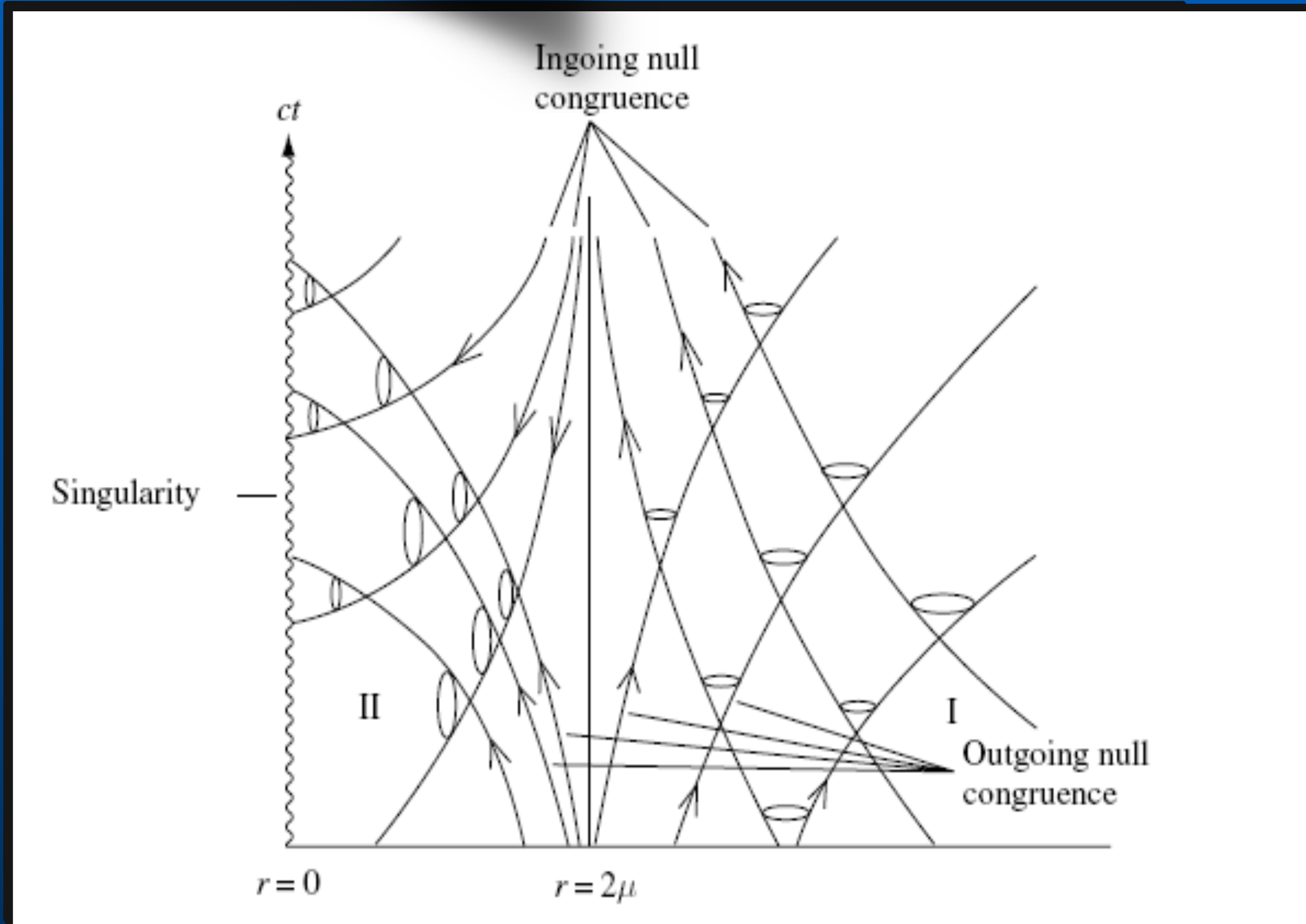
Structure of a Schwarzschild black hole



$$r_{\text{Schw}} \sim 3\left(\frac{M}{M_\odot}\right)\,\text{km},$$

An essential singularity occurs when $g_{tt} \to \infty$

Structure of a Schwarzschild black hole

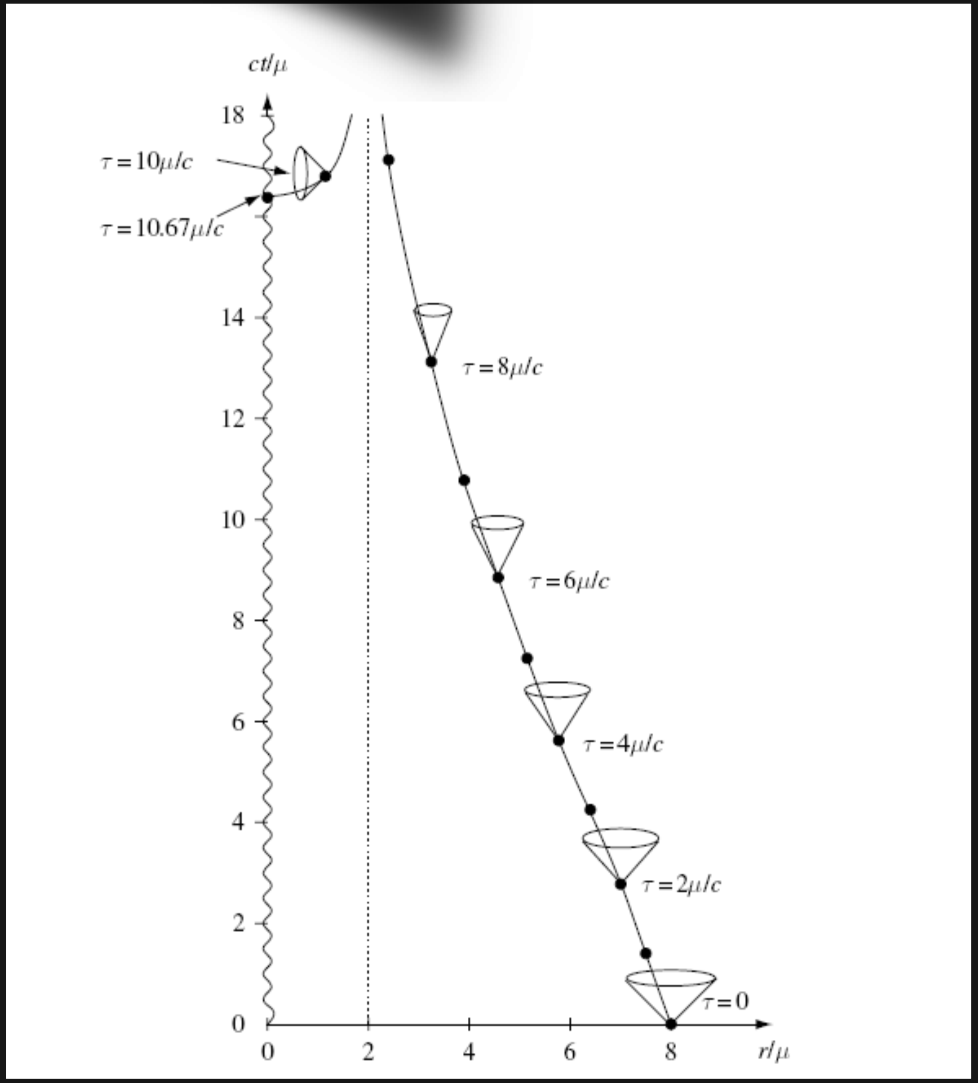


$$ds^2 = 0.$$

$$\frac{dr}{dt} = \pm\left(1 - \frac{2GM}{r}\right),$$

$$r \to 2GM,\ dr/dt \to 0$$

Structure of a Schwarzschild black hole



Structure of a Schwarzschild black hole

The singularity at the Schwarzschild radius is only apparent, since it can be removed through a coordinate change. Let us consider, for instance, Eddignton-Finkelstein coordinates:

$$r_* = r + \frac{2GM}{c^2} \log \left| \frac{r - 2GM/c^2}{2GM/c^2} \right|.$$

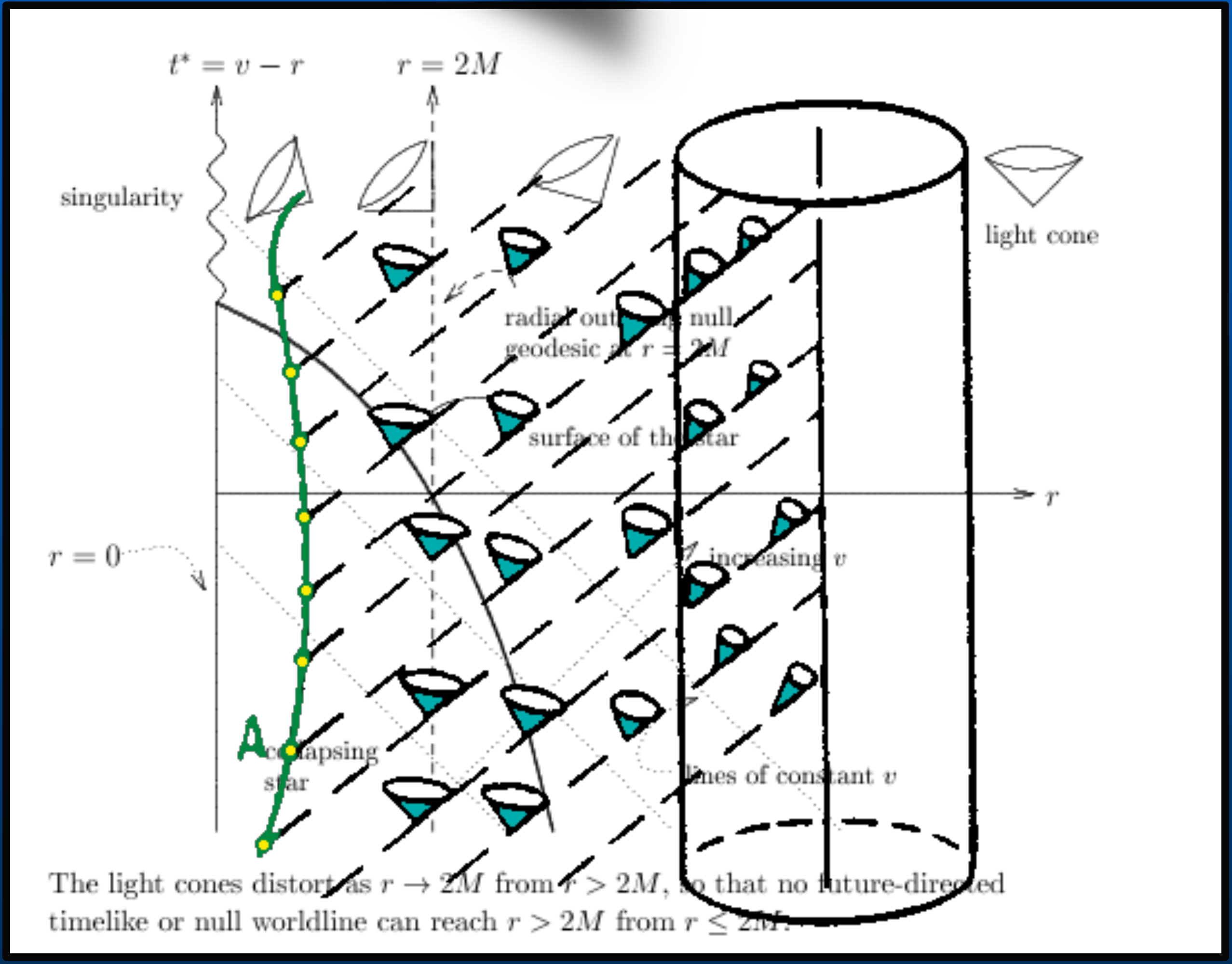
$$v = ct + r_*.$$

Null rays satisfy dv=0. The new coordinate v can be used as a time coordinate. The metric can be rewritten as:

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right) dv^2 - 2dr\,dv - r^2 d\Omega^2,$$

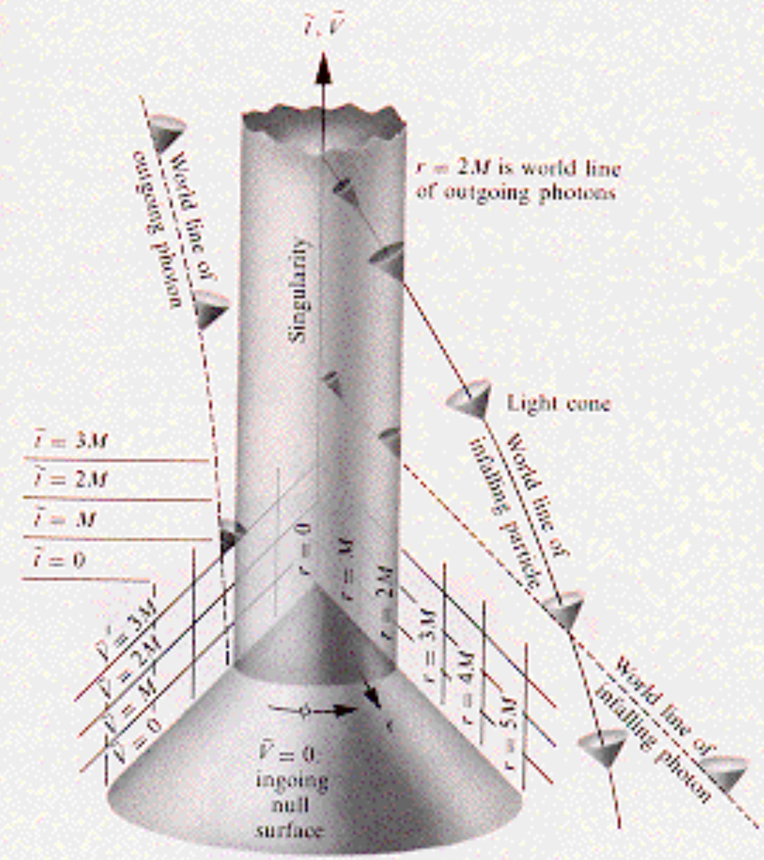
$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2.$$

Structure of a Schwarzschild black hole



Event horizon in Schwarzschild spacetime

Collapse in Eddington-Finkelstein coordinates



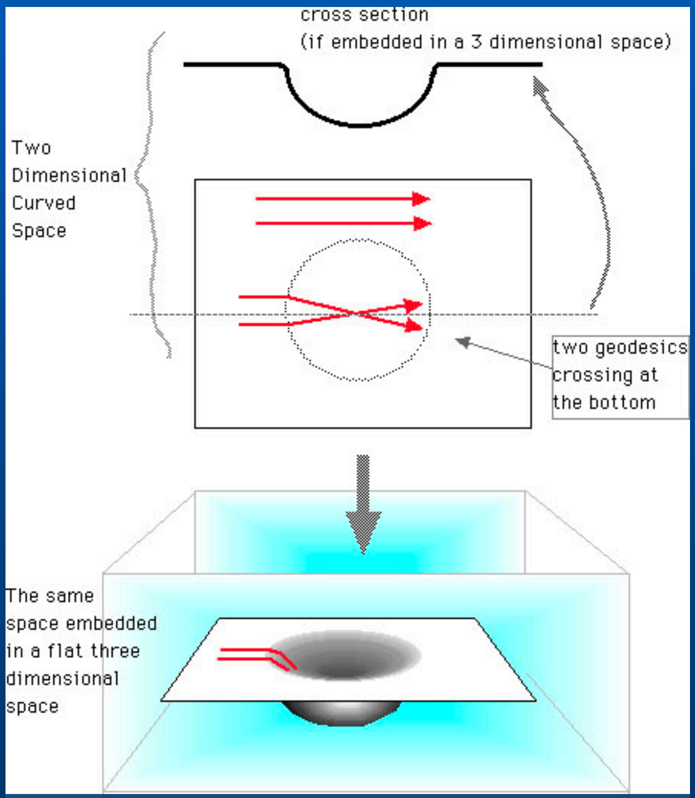
Ingoing Finkelstein coordinates (one rotational degree of freedom is suppressed; i.e., θ is set equal to $\pi/2$). Surfaces of constant $\tilde{V}$, being ingoing null surfaces, are plotted on a 45-degree slant, just as they would be in flat spacetime. Equivalently, surfaces of constant

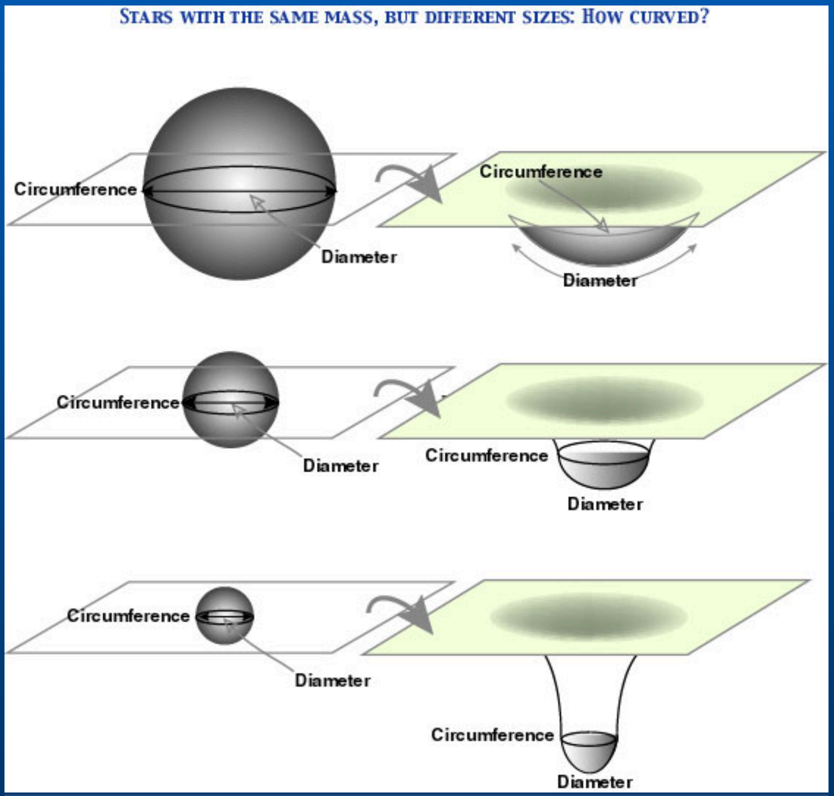
$$\tilde{t} \equiv \tilde{V} - r = t + 2M \ln |r/2M - 1|$$

are plotted as horizontal surfaces.

Embedding

In an embedding diagram the curvature of a two dimensional surface can be viewed by placing it in a flat three-dimensional space. In general, these diagrams come from the elimination of a third dimension in order to show the spatial cross section at a given time of a solution to Einstein's equations



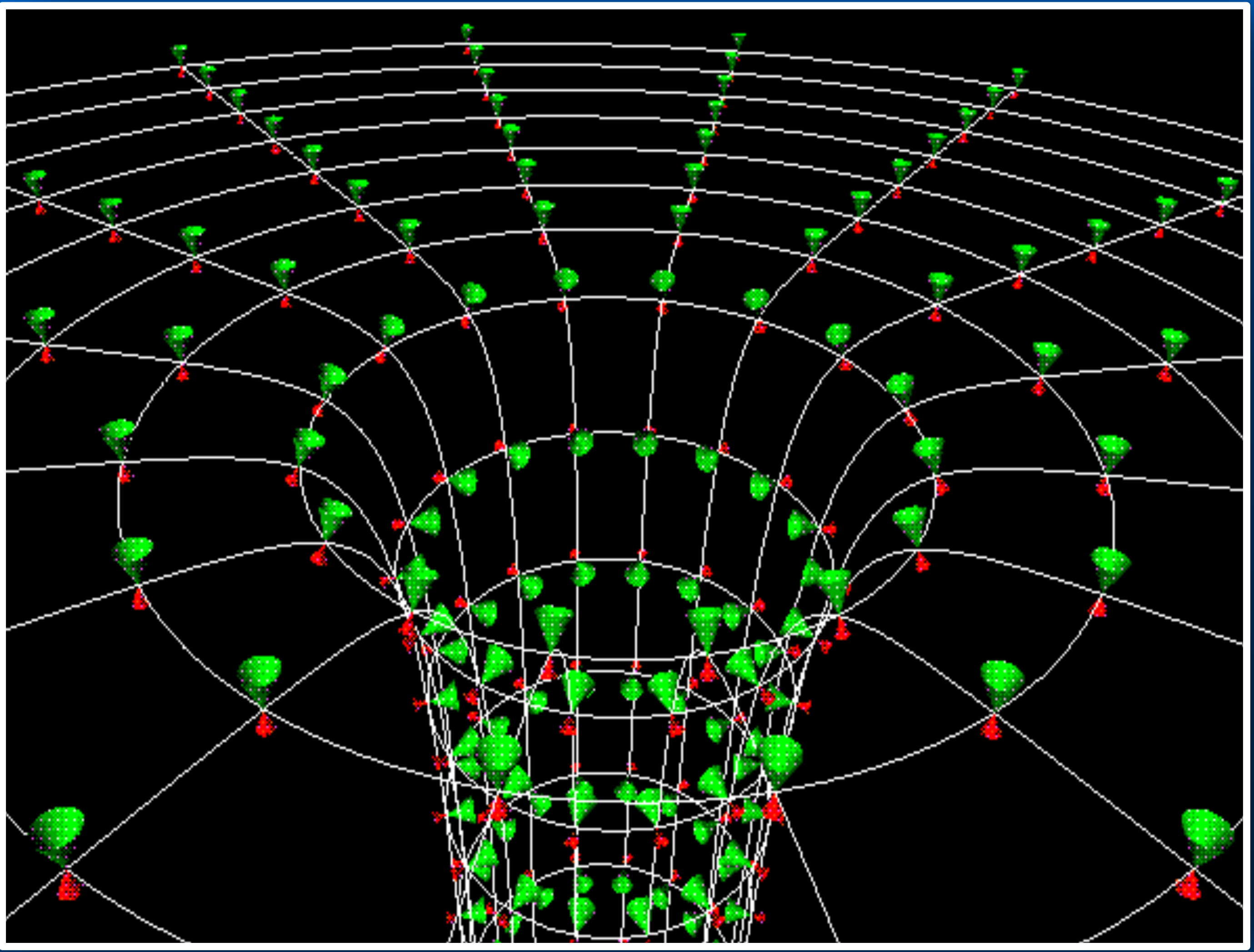


$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{48\mu^2}{r^6},$$

$$\mu \equiv GM/c^2.$$

$$C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta} = 12\left(\frac{R_S}{r^3}\right)^2 = \frac{48M^2G^2}{r^6c^4}$$

Embedding



$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{48\mu^2}{r^6},$$

$$\mu \equiv GM/c^2.$$

$$C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta} = 12\left(\frac{R_S}{r^3}\right)^2 = \frac{48M^2G^2}{r^6c^4}$$

Birkoff's theorem

If we consider the isotropic but *not static* line element,

$$ds^2 = c^2 A(r,\ t)dt^2 - B(r,\ t)dr^2 - r^2 d\Omega^2,$$

and substitute into the Einstein empty-space field equations $R_{\mu\nu} = 0$ to obtain the functions $A(r,\ t)$ and $B(r,\ t)$, the result would be exactly the same:

$$A(r,\ t) = A(r) = \left(1 - \frac{2GM}{rc^2}\right),$$

and

$$B(r,\ t) = B(r) = \left(1 - \frac{2GM}{rc^2}\right)^{-1}.$$

Birkoff's theorem

The space-time geometry outside a general spherically symmetric matter distribution is the Schwarzschild geometry.

Birkhoff's theorem implies that strictly radial motions do not perturb the spacetime metric. In particular, a pulsating star, if the pulsations are strictly radial, does not produce gravitational waves.

The converse of Birkhoff's theorem is not true, i.e.,

If the region of space-time is described by the Schwarzschild metric, then the matter distribution that is the source of the metric does not need to be spherically symmetric.

Causal structure of space-time

Definition. *A causal curve in a space-time* $(M,\ g_{\mu\nu})$ *is a curve that is non space-like, that is, piecewise either time-like or null (light-like).*

We say that a given space-time $(M,\ g_{\mu\nu})$ is *time-orientable* if we can define over M a smooth non-vanishing time-like vector field.

Definition. *If* $(M,\ g_{\mu\nu})$ *is a time-orientable space-time, then* $\forall p \in M$, *the causal future of* p, *denoted* $J^+(p)$, *is defined by:*

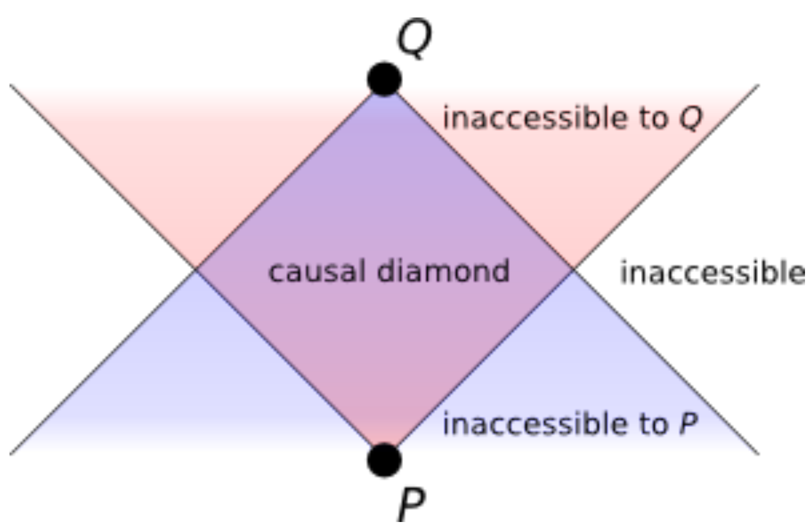
$$J^+(p) \equiv \{q \in M | \exists\ a\ future-directed\ causal\ curve\ from\ p\ to\ q\}\,.$$

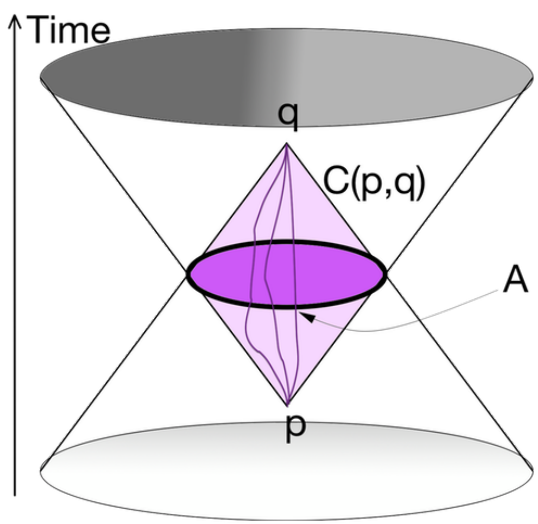
Causal structure of space-time

Similarly,

Definition. *If $(M,\ g_{\mu\nu})$ is a time-orientable space-time, then $\forall p \in M$, the causal past of p, denoted $J^-(p)$, is defined by:*

$$J^-(p) \equiv \{q \in M | \exists \text{ a past} - \text{directed causal curve from } p \text{ to } q\}\,.$$





Causal structure of space-time

The causal future and past of any set $S \subset M$ are given by:

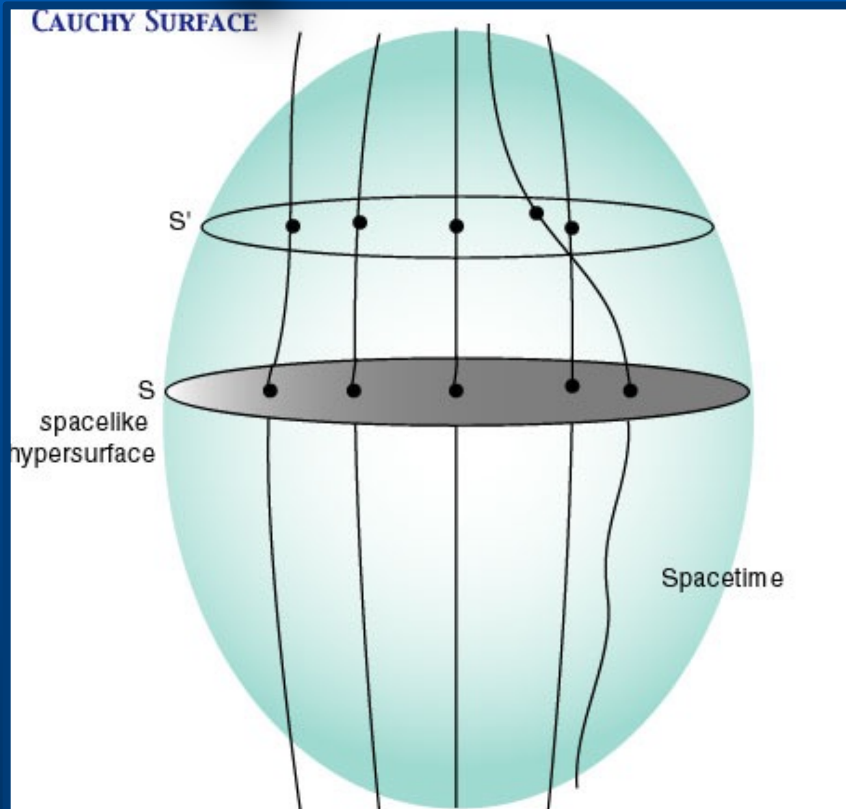
$$J^+(S) = \bigcup_{p \in S} J^+(P)$$

and,

$$J^-(S) = \bigcup_{p \in S} J^-(P).$$

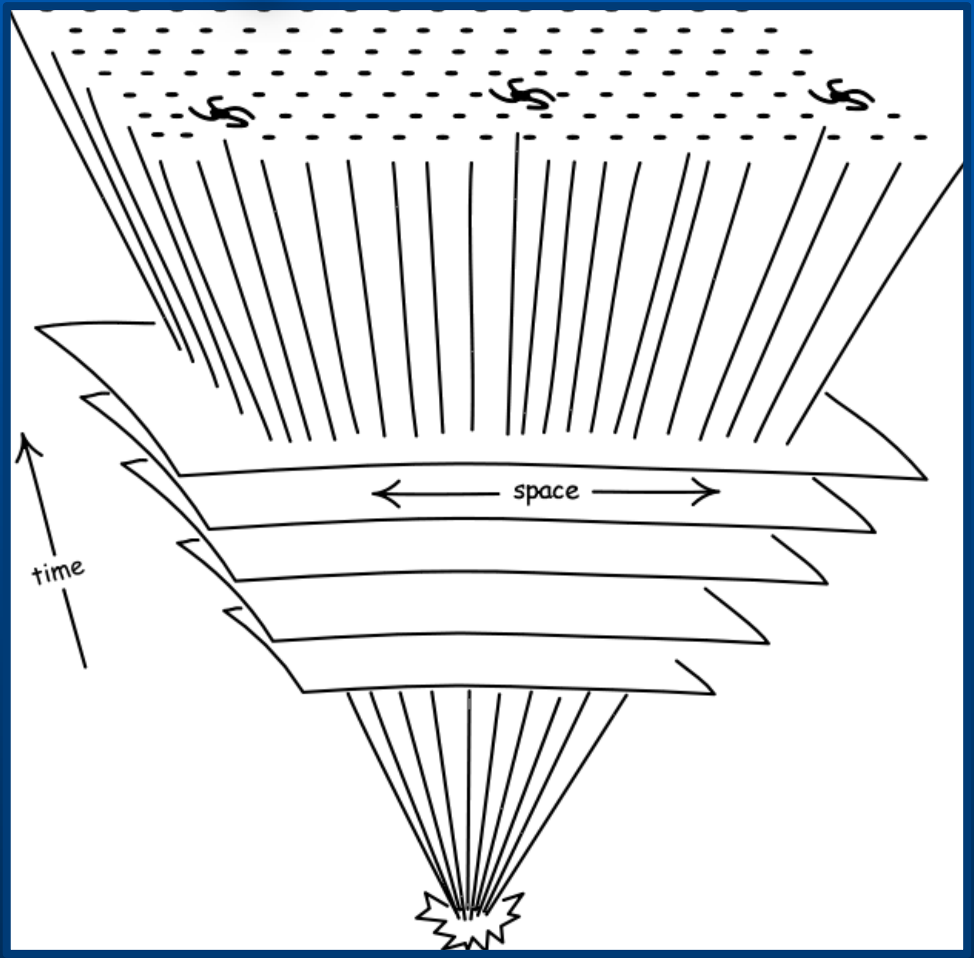
A set S is said *achronal* if no two points of S are time-like related. A Cauchy surface is an achronal surface such that every non space-like curve in M crosses it once, and only once, S. A space-time $(M, g_{\mu\nu})$ is *globally hyperbolic* if it admits a space-like hypersurface $S \subset M$ which is a Cauchy surface for M.

Causal structure of space-time



S is called **Cauchy surface** if ***every*** world-line (timelike curve) without endpoint intersects once and only once the hypersurface S.
If S and S' are a Cauchy surface, the events on S' are determined by those on S, if the law governing this spacetime is deterministic.
General relativity allows non-existence of Cauchy surfaces in certain cases.

Causal structure of space-time



Causal structure of space-time

Causal relations are invariant under conformal transformations of the metric. In this way, the space-times $(M, g_{\mu\nu})$ and $(M, \widetilde{g}_{\mu\nu})$, where $\widetilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, with Ω a non-zero C^r function, have the same causal structure.

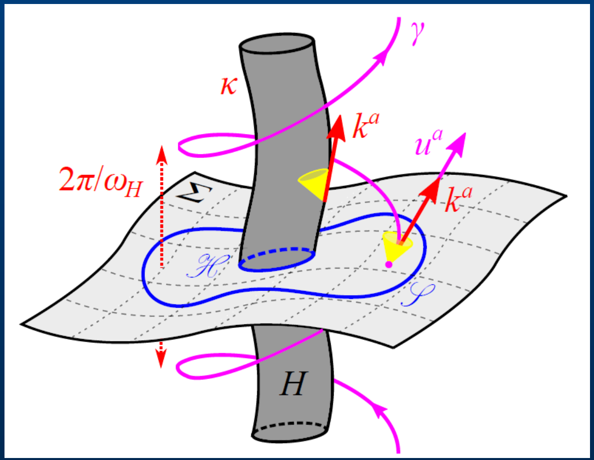
Formal definition of black hole

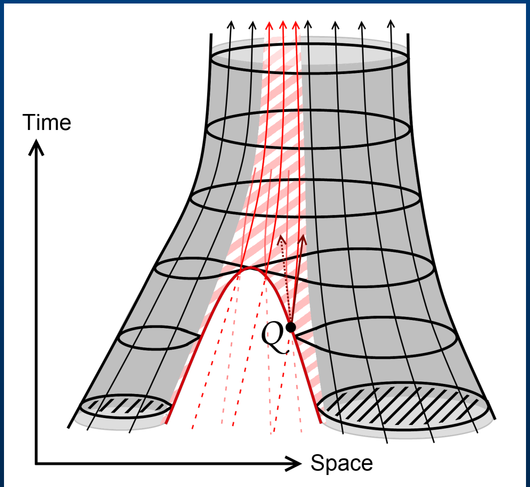
Let us now consider a space-time where all null geodesics that start in a region $\mathcal{J}^-$ end at $\mathcal{J}^+$. Then, such a space-time, $(M,\ g_{\mu\nu})$, is said to contain a *black hole* if M *is not* contained in $J^-(\mathcal{J}^+)$. In other words, there is a region from where no null geodesic can reach the *asymptotic flat*future space-time, or, equivalently, there is a region of M that is causally disconnected from the global future. The *black hole region*, BH, of such space-time is $BH = [M - J^-(\mathcal{J}^+)]$, and the boundary of BH in M, $H = \dot{J}^-(\mathcal{J}^+) \bigcap M$, is the *event horizon*

Formal definition of black hole

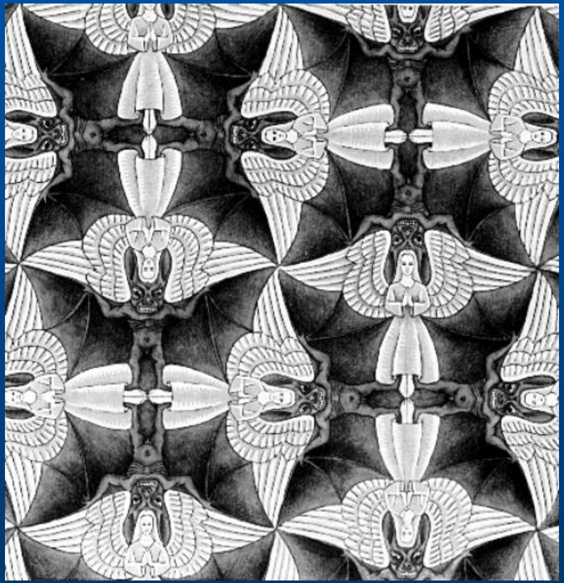
A black hole is conceived as a space-time *region*, i.e. what characterizes the black hole is its metric and, consequently, its curvature.

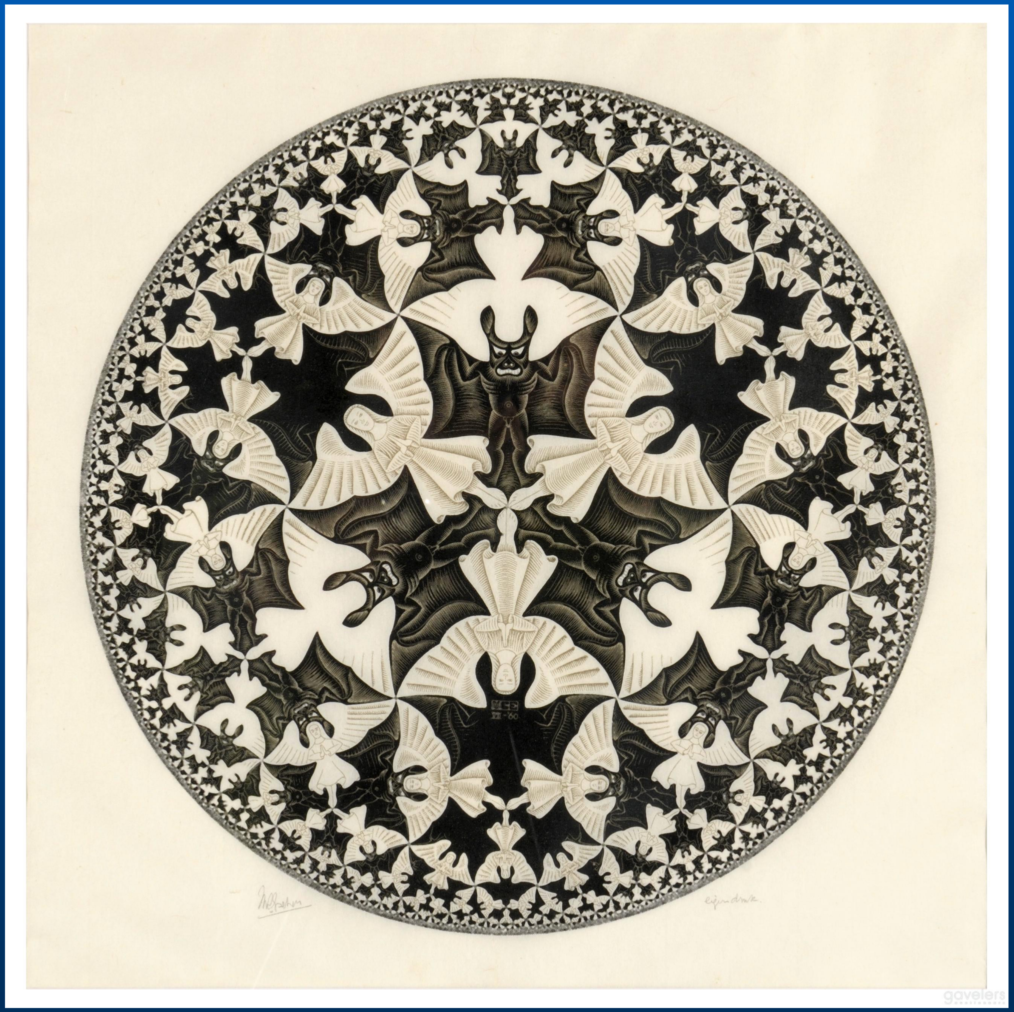
What is peculiar of this space-time region is that it is causally disconnected from the rest of the space-time: no events in this region can make any influence on events outside the region. Hence the name of the boundary, event horizon: events inside the black hole are separated from events in the global external future of space-time. The events in the black hole, nonetheless, as all events, are causally determined by past events. A black hole does not represent a breakdown of classical causality.





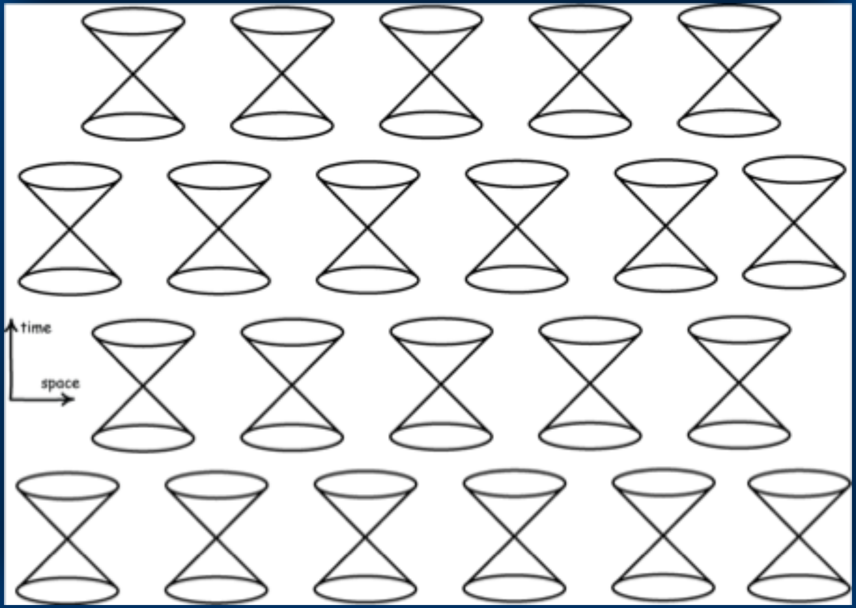
Conformal diagrams

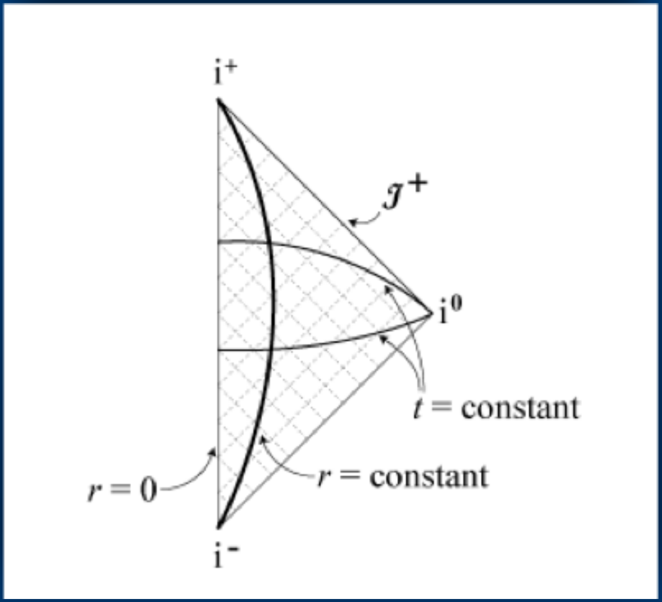




Conformal diagrams

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}.$$

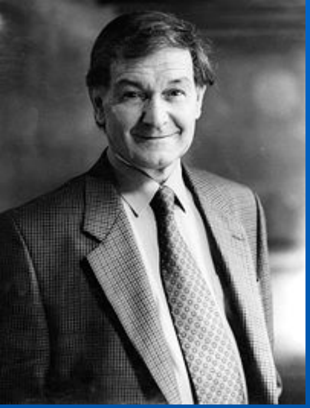




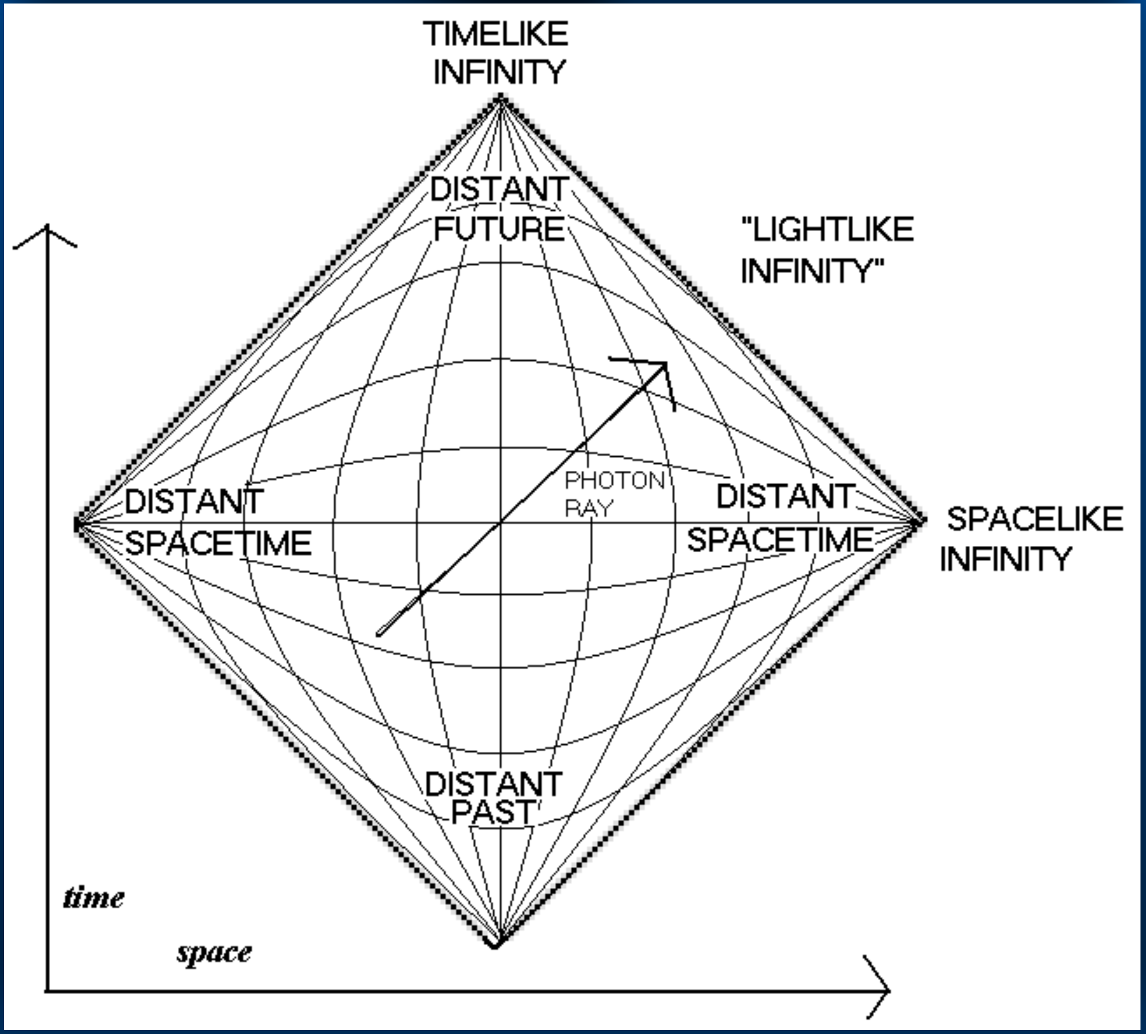
$$v = t + r, \quad u = t - r,$$

$$ds^2 = -du\,dv + \tfrac{1}{4}(u-v)^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$\Omega^2 = (1+v^2)^{-1}(1+u^2)^{-1},$$



Penrose-Carter diagram

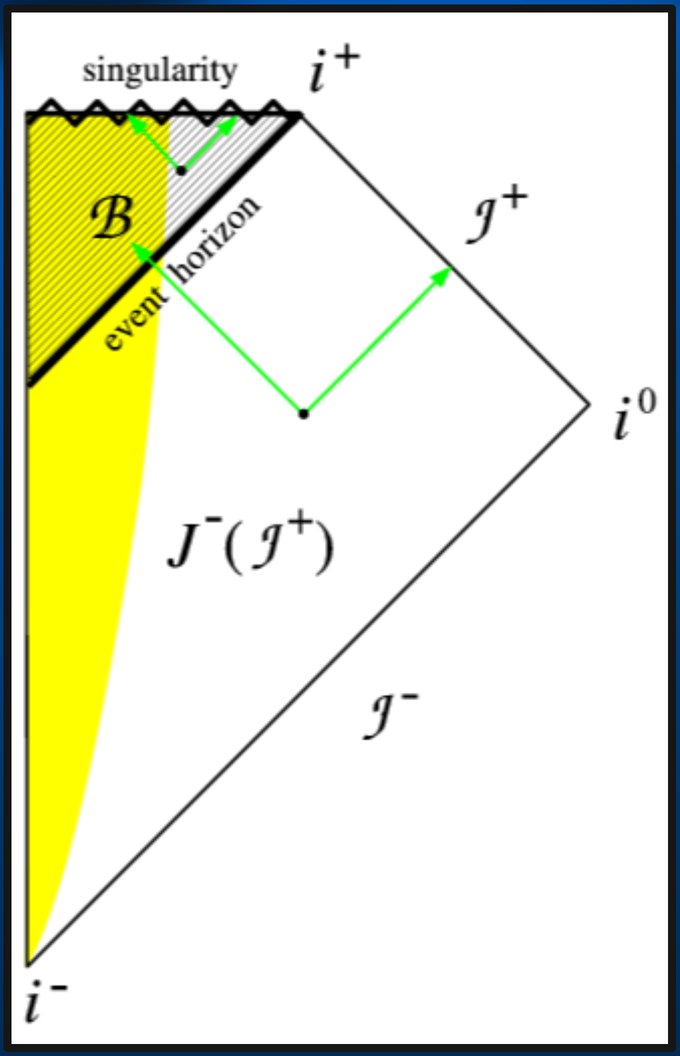


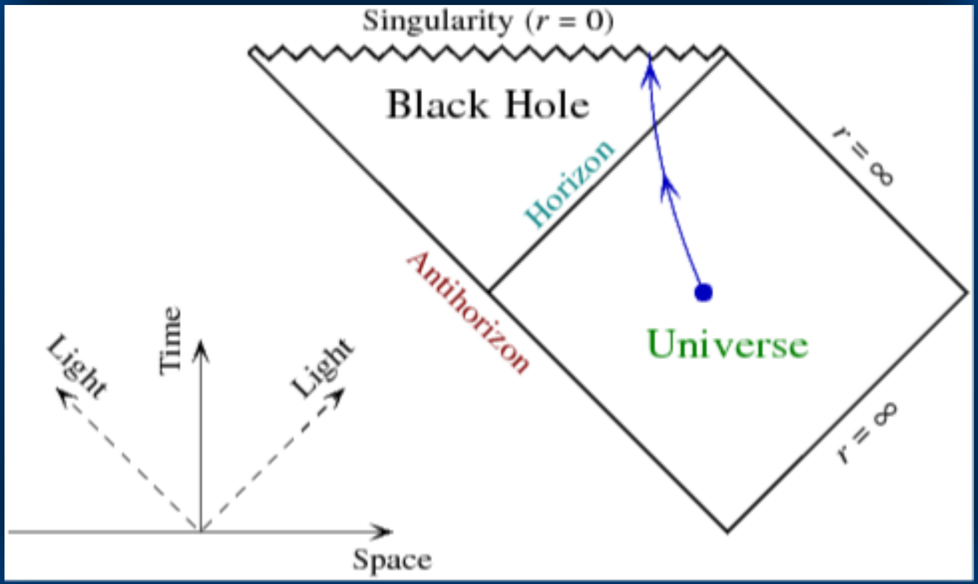
The conformal factor is chosen such that the entire infinite spacetime is transformed into a Penrose diagram of finite size.

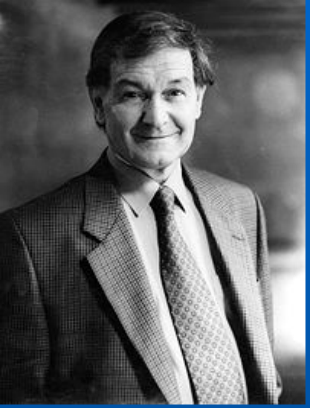
$$\tan(u \pm v) = x \pm t$$

Minkowski space-time

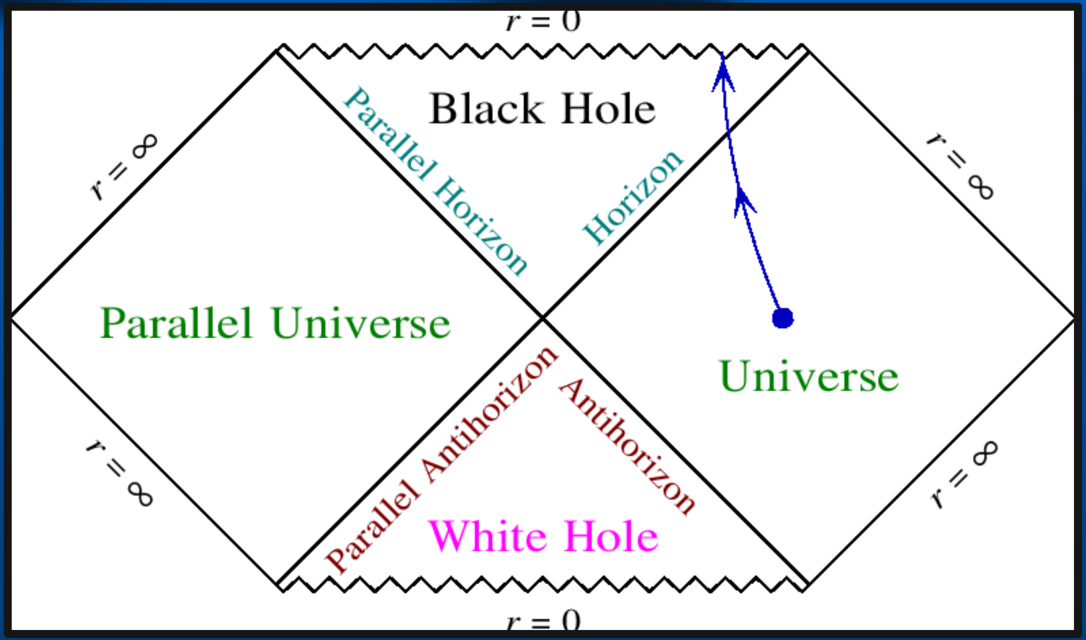
Formal definition of black hole

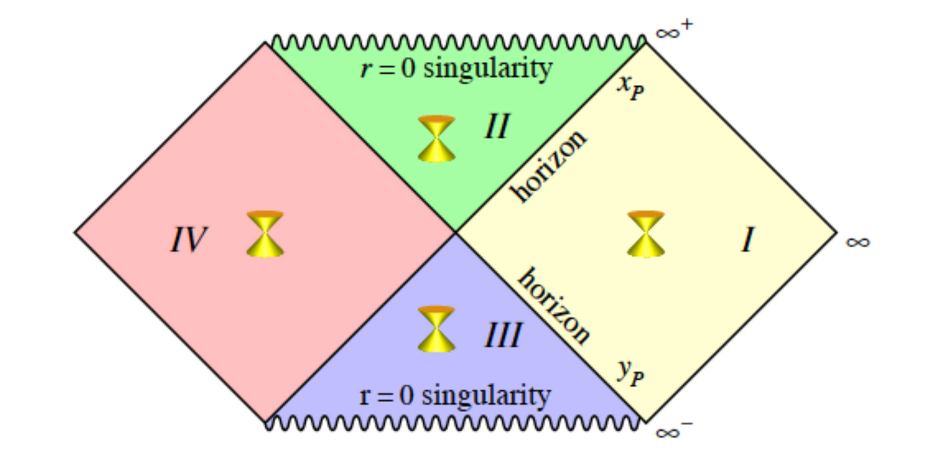






Penrose-Carter diagram





Schwarzschild space-time

Kruskal-Szekeres coordinates

$$
\begin{aligned}
u &= \left(\frac{r}{r_{\text{Schw}}} - 1\right)^{1/2} e^{\frac{r}{2r_{\text{Schw}}}} \cosh\left(\frac{ct}{2r_{\text{Schw}}}\right), \\
v &= \left(\frac{r}{r_{\text{Schw}}} - 1\right)^{1/2} e^{\frac{r}{2r_{\text{Schw}}}} \sinh\left(\frac{ct}{2r_{\text{Schw}}}\right), \\
& \qquad \text{if} \quad r > r_{\text{Schw}},
\end{aligned}
$$

$$
\begin{aligned}
u &= \left(1 - \frac{r}{r_{\text{Schw}}}\right)^{1/2} e^{\frac{r}{2r_{\text{Schw}}}} \sinh\left(\frac{ct}{2r_{\text{Schw}}}\right), \\
v &= \left(1 - \frac{r}{r_{\text{Schw}}}\right)^{1/2} e^{\frac{r}{2r_{\text{Schw}}}} \cosh\left(\frac{ct}{2r_{\text{Schw}}}\right), \\
& \qquad \text{if} \quad r < r_{\text{Schw}}.
\end{aligned}
$$

Kruskal-Szekeres coordinates

The line element in the Kruskal-Szekeres coordinates is completely regular, except at $r = 0$:

$$ds^2 = \frac{4r_{\rm Schw}^3}{r} e^{\frac{r}{r_{\rm Schw}}} \left(dv^2 - du^2\right) - r^2 d\Omega^2.$$

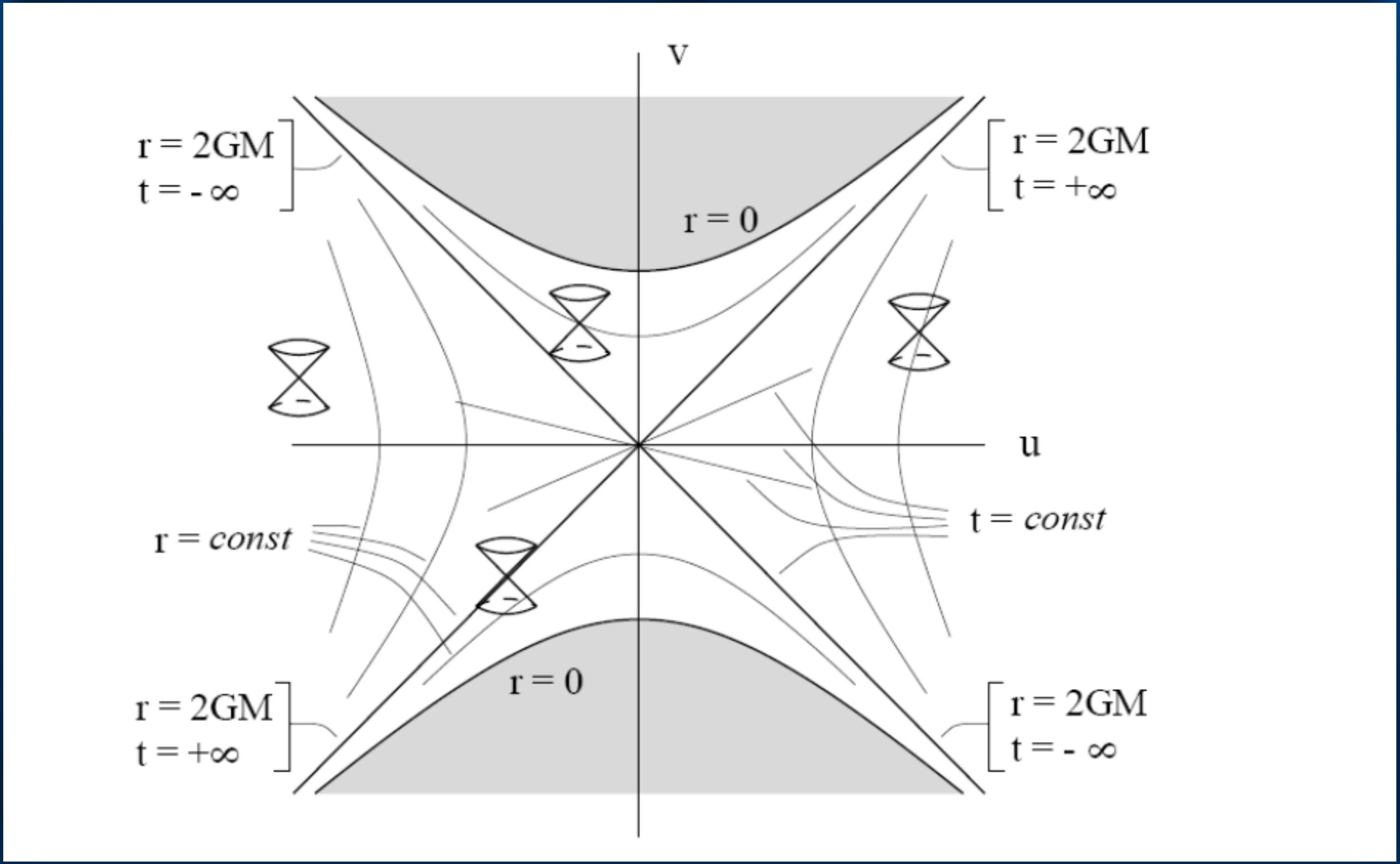
The curves at $r =$ constant are hyperbolic and satisfy:

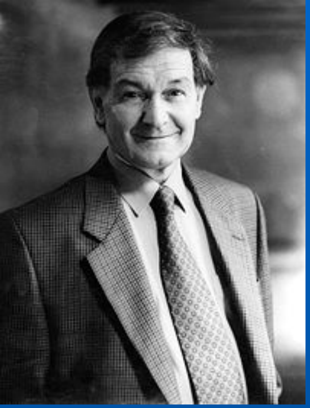
$$u^2 - v^2 = \left(\frac{r}{r_{\rm Schw}} - 1\right)^{1/2} e^{\frac{r}{r_{\rm Schw}}},$$

the curves at $t =$ constant are straight lines that pass through the origin:

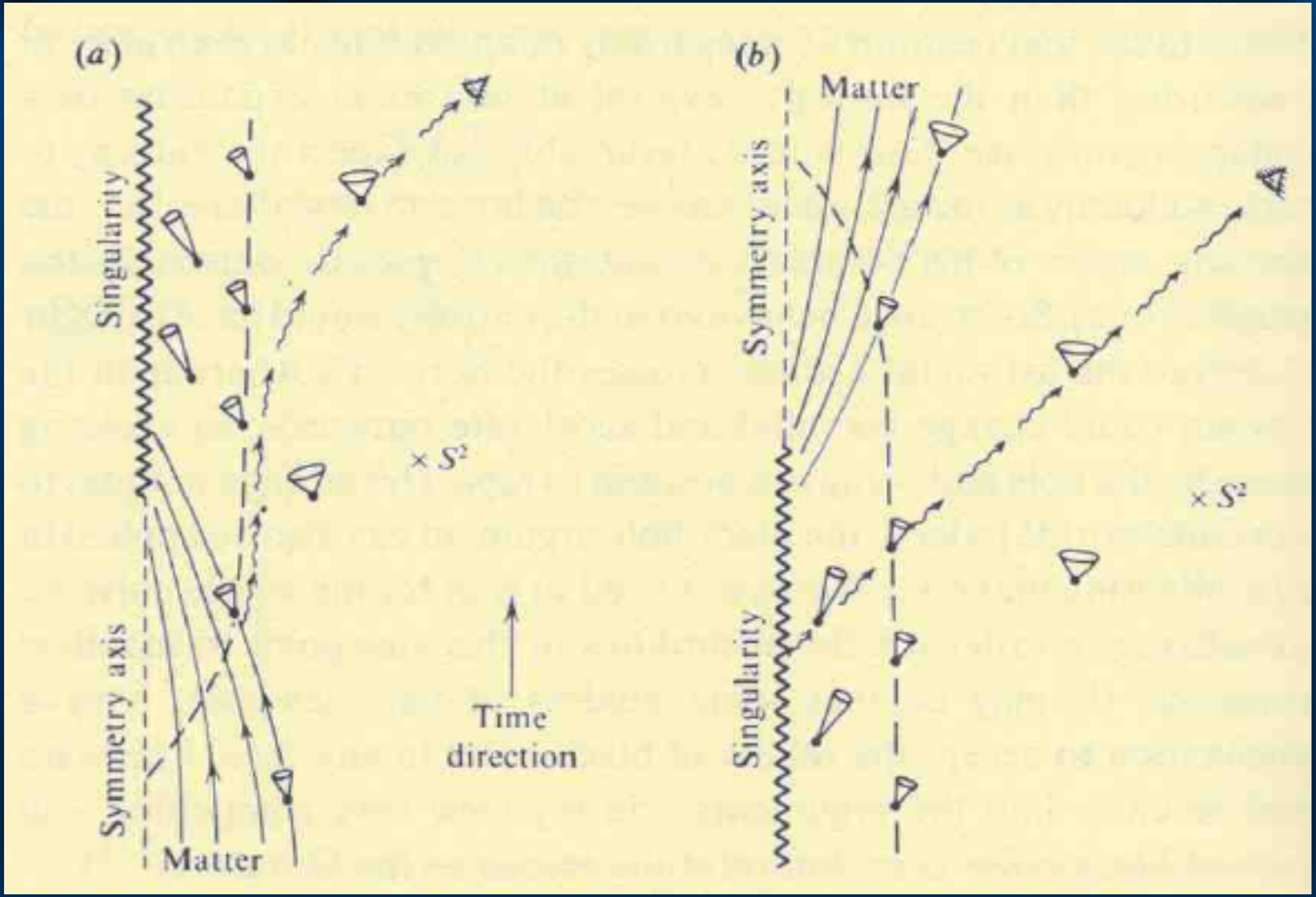
$$\begin{aligned} \frac{u}{v} &= \tanh \frac{ct}{2r_{\rm Schw}}, \quad r < r_{\rm Schw}, \\ \frac{u}{v} &= \coth \frac{ct}{2r_{\rm Schw}}, \quad r > r_{\rm Schw}. \end{aligned}$$

Kruskal-Szekeres coordinates





White hole



(a)
Singularity
Symmetry axis
Matter
× S²
Time direction
(b)
Matter
Symmetry axis
Singularity
× S²

Horizons in General Relativity

A null geodesic is a curve on the spacetime manifold which has null tangent l_a (i.e., $l^a l_a = 0$) and satisfies the geodesic equation

$$l^b \nabla_b l^a = 0$$

The 2-metric orthogonal to l_a is is the projection tensor onto the hyperplanes orthogonal to the null direction.

$$h_{ab} \equiv g_{ab} + l_a n_b + l_b n_a \,.$$

$$l^c n_c = -1$$

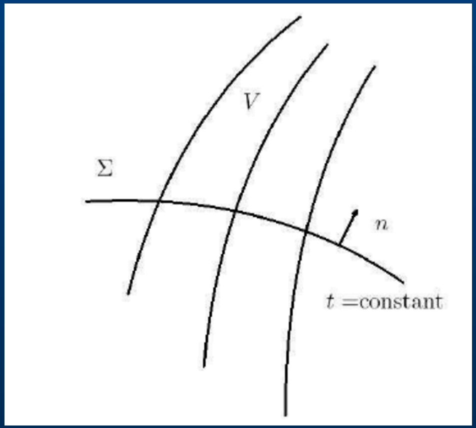
$$h_{ab}\, l^a = h_{ab}\, l^b = 0\,,$$

$$h^a{}_a = 2\,,$$

$$h^a{}_c\, h^c{}_b = h^a{}_b\,.$$

Congruence

Let O be an open region of the spacetime manifold; a ***congruence of curves*** in O is a family of curves such that through every point of O passes one and only one curve of the family. The tangents to these curves define a vector field on O (and, conversely, every continuous vector field in O generates a congruence of curves, those to which the vector field is tangent). If the field of tangents is smooth, we say that the congruence is smooth. In particular, we can consider a congruence of null geodesics with tangents l_a in the open region O.



$$\theta = \nabla_c l^c$$

$$B_{ab} \equiv \nabla_b l_a ,$$

$$\theta_l = h^{ab} \nabla_a l_b$$

Expansion

$$\theta_{ab} \equiv \frac{\theta}{2} h_{ab}$$

Shear

$$\sigma_{ab} \equiv B_{(ab)} - \frac{\theta}{2} h_{ab}$$

Vorticity

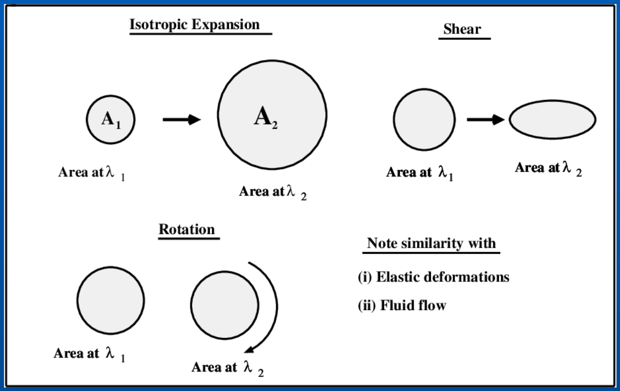
$$\omega_{ab} \equiv B_{[ab]}$$

$$\sigma^2 \equiv \sigma_{ab} \sigma^{ab} , \qquad \omega^2 \equiv \omega_{ab} \omega^{ab}$$

$$\sigma^a{}_a = \omega^a{}_a = 0.$$

The expansion, shear, and vorticity tensors are purely transversal:

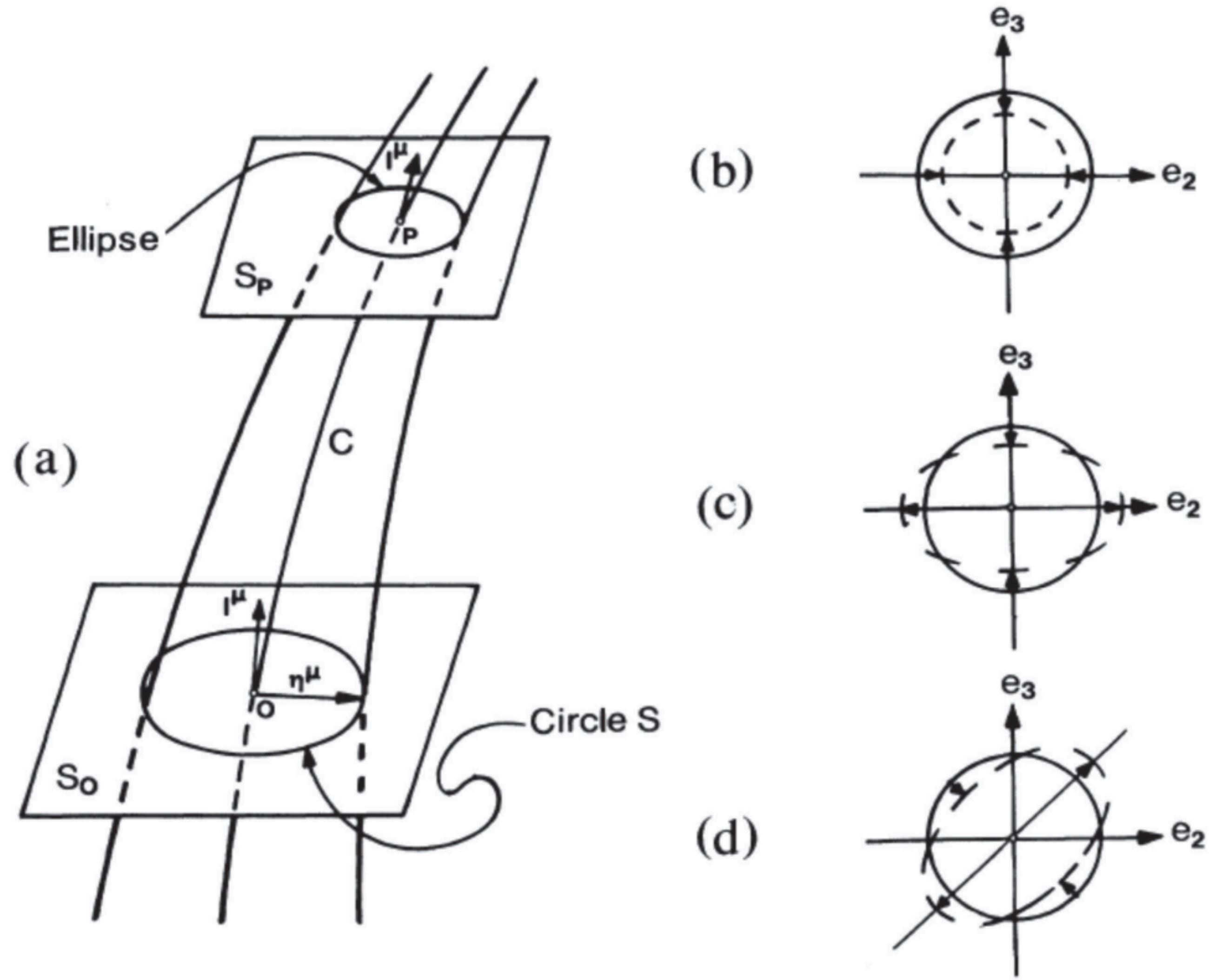
$$\theta_{ab}\, l^a = \theta_{ab}\, l^b = 0\,,$$
$$\sigma_{ab}\, l^a = \sigma_{ab}\, l^b = 0\,,$$
$$\omega_{ab}\, l^a = \omega_{ab}\, l^b = 0\,,$$



The propagation of the expansion along a null geodesics ruled by the *Raychaudhuri equation*:

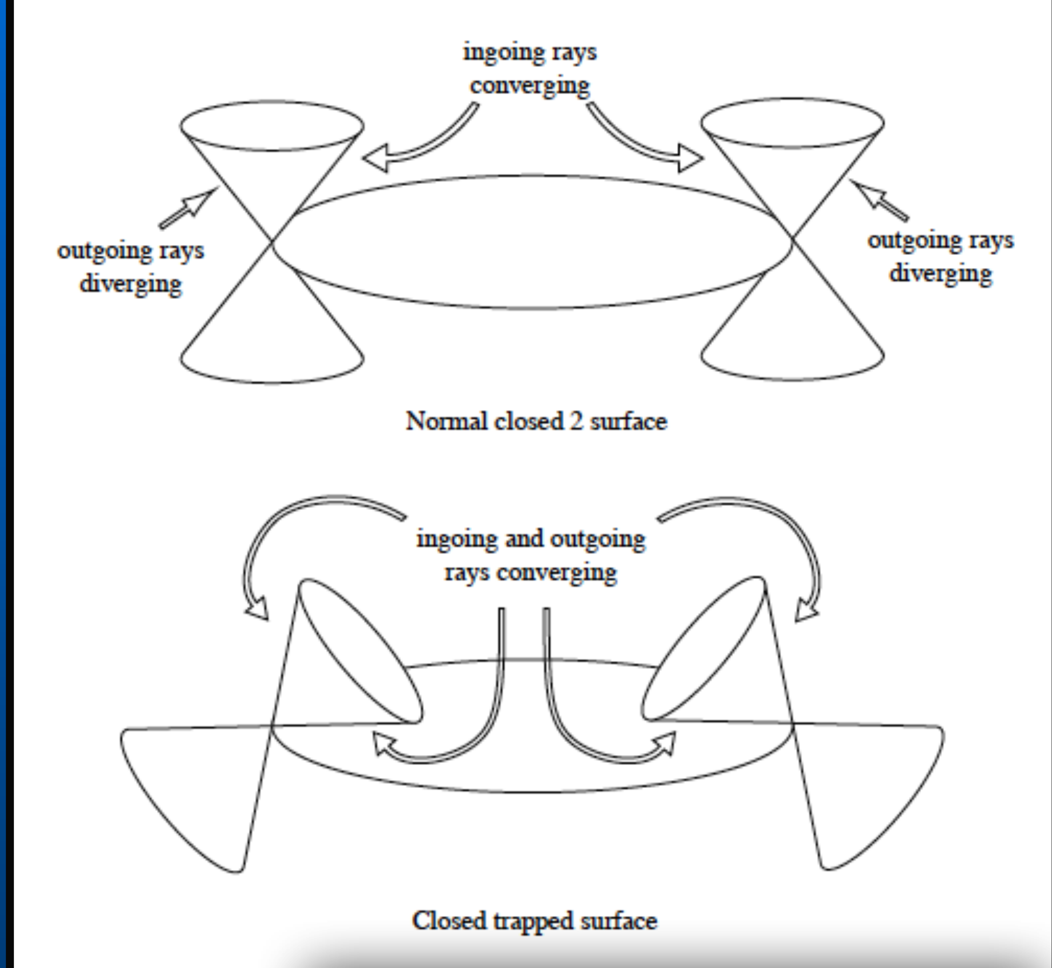
$$\longrightarrow \quad \frac{d\theta}{d\lambda} = -\frac{\theta^2}{2} - \sigma^2 + \omega^2 - R_{ab} l^a l^b$$

Equation of Raychaudhuri



Ellipse
l^μ
P
S_P
(a)
C
l^μ
η^μ
O
Circle S
S_O
(b)
e_3
e_2
(c)
e_3
e_2
(d)
e_3
e_2

Trapped surfaces



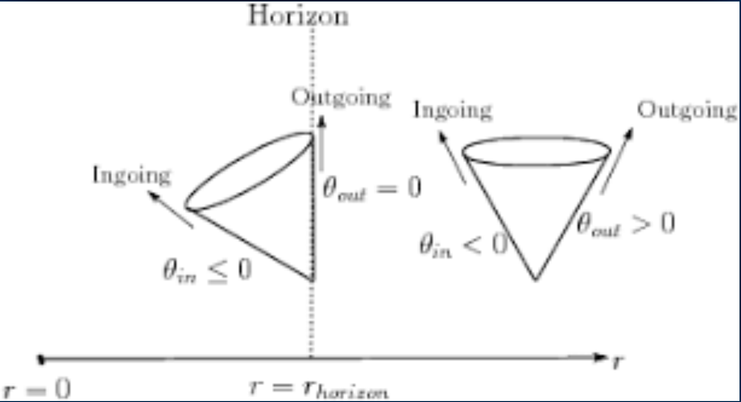
A normal surface corresponds to

$$\theta_l > 0 \text{ and } \theta_n < 0$$

A trapped surface corresponds to:

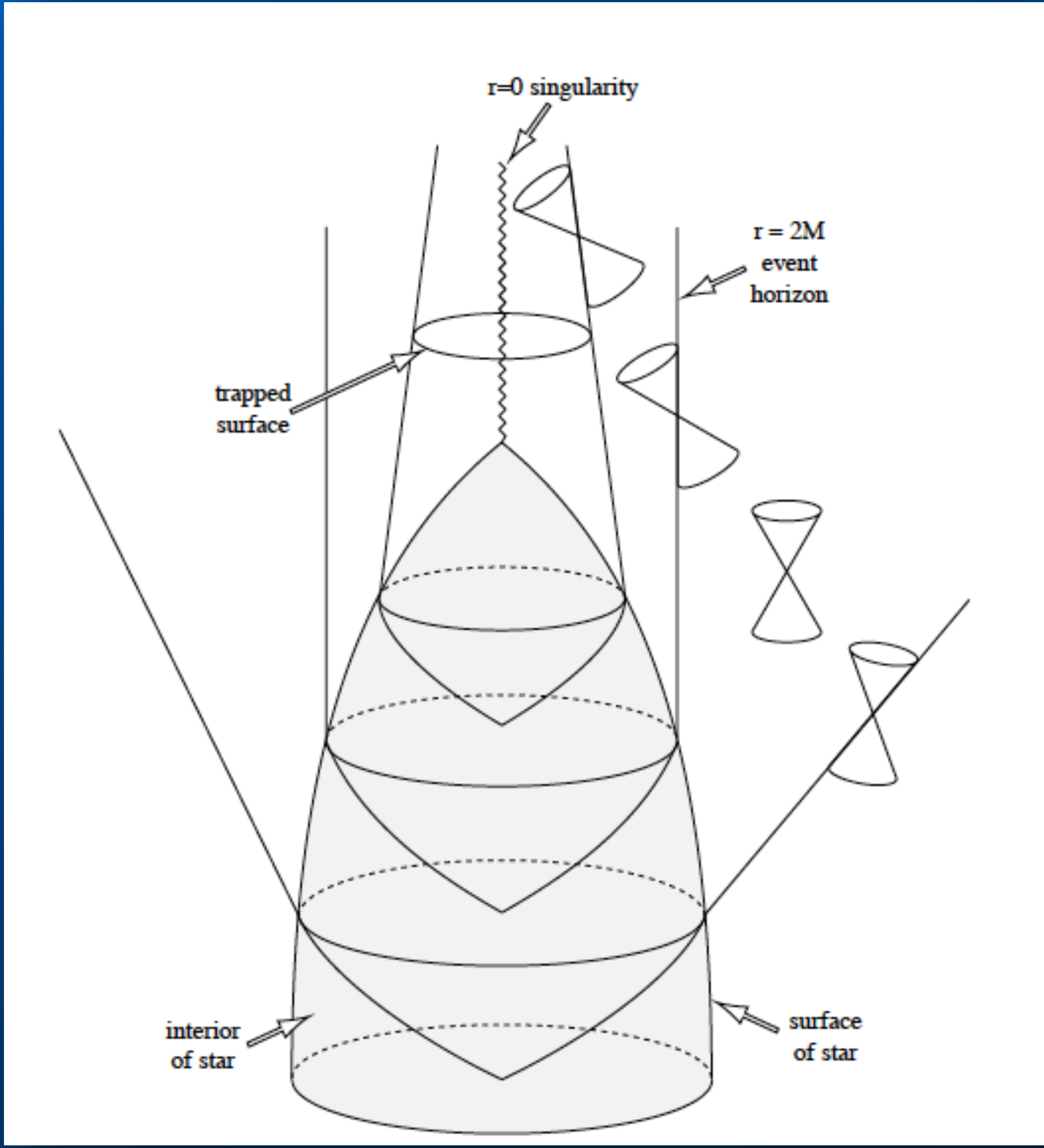
$$\theta_l < 0 \text{ and } \theta_n < 0.$$

In this case, the outgoing, in addition to the usual ingoing, future-directed null rays are converging instead of diverging—light propagating outward is dragged back by strong gravity.



Black hole

Trapped surfaces seem to be essential features in the black hole concept and *notions of "horizon" of practical utility will be identified with the boundaries of spacetime regions which contain trapped surfaces.*

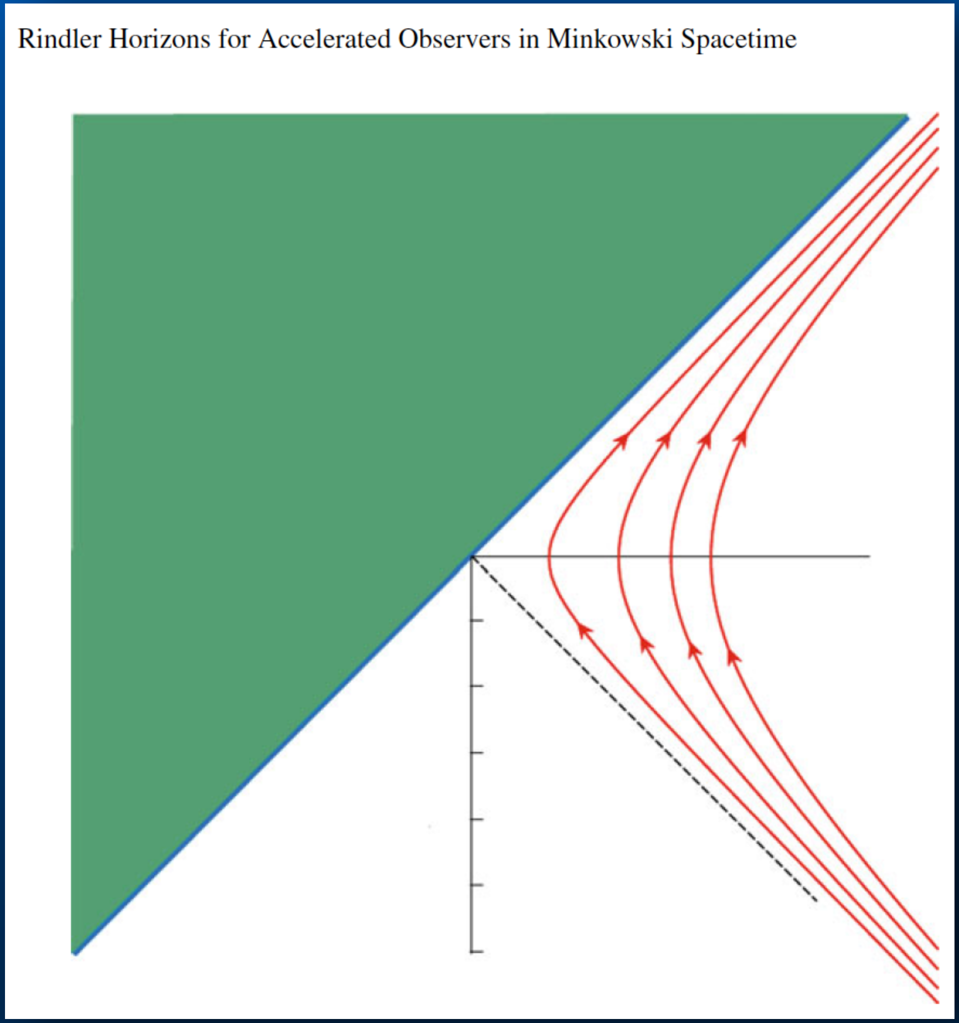


Rindler horizons

$$ds^2 = -(\alpha x)^2 dt^2 + dx^2 + dy^2 + dz^2$$

A family of hyperbolae parametrized by the constant acceleration *a* are called hyperbolic motions or worldlines of Rindler observers.

$$x^2 - t^2 = \left(\frac{1}{a}\right)^2$$



The location of the event horizon depends on the uniformly accelerated observer: different accelerated observers will determine different acceleration horizons.

Event horizons

An event horizon is a *connected component of the boundary of the causal past of future null infinity*.

This definition embodies the most peculiar feature of a black hole, i.e., the horizon is a causal boundary which separates a region from which nothing can come out to reach a distant observer from a region in which signals can be sent out and eventually arrive to this observer. An event horizon is generated by the null geodesics which fail to reach infinity and, therefore (provided that it is smooth) is always a null hypersurface .

In black hole research and in astrophysics the concept of event horizon is implicitly taken to define the concept of static or stationary black hole itself.

Since to define and locate an event horizon one must know all the future history of spacetime (one must know all the geodesics which do reach null infinity and, tracing them back, the boundary of the region from which they originate), *an event horizon is a globally defined concept*. To state that an event horizon has formed requires knowledge of the spacetime outside our future light cone, which is impossible to achieve (unless, of course, the spacetime is stationary and the black hole has existed forever—then nothing changes and by knowing the state of the world now one knows it forever).

Because of its global nature, an **event horizon is not a practical notion to work with**, and it is nearly impossible to locate precisely an event horizon in a general dynamical situation.

In practice, astrophysical black holes did not exist forever but formed in a highly dynamical process of gravitational collapse. Numerical relativity codes are written to follow a gravitational collapse, the merger of a binary system, or other dynamical situations ending in a black hole, and they crash at some point. It is clearly impossible to follow the evolution of a system all the way to future null infinity.

Numerical relativists routinely use marginally trapped surfaces as proxies for event horizons.

Apparent horizons

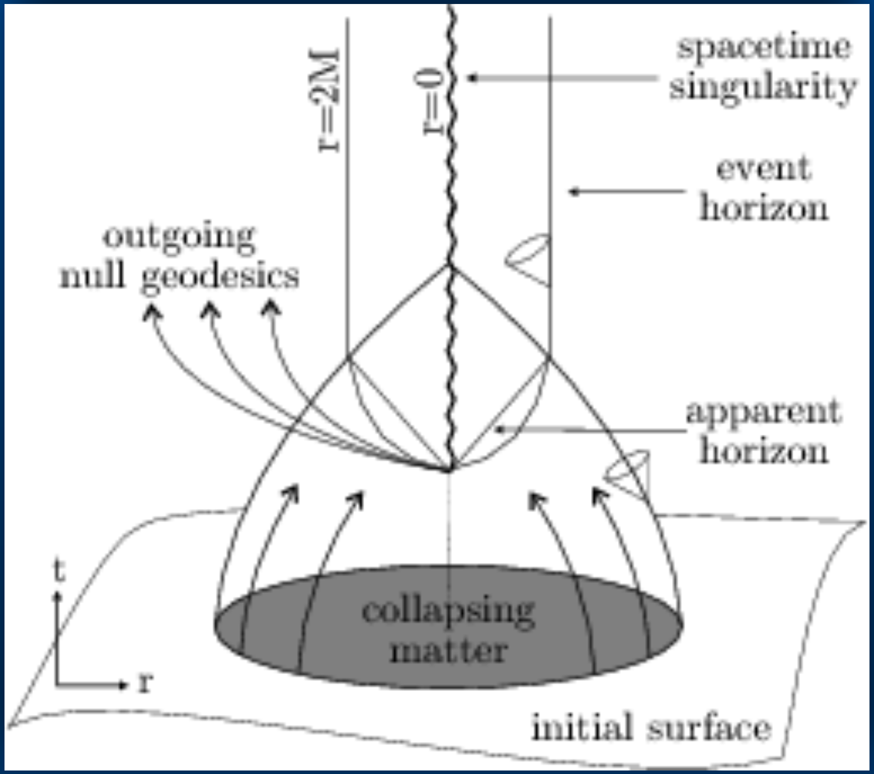
A future apparent horizon is the closure of a surface (usually a 3-surface) which is foliated by marginal surfaces. The future apparent horizon is a surface defined by the conditions on the time slicings

$$\theta_l = 0\,,$$
$$\theta_n < 0\,,$$

These are the expansions of the future-directed outgoing and ingoing null geodesic congruences, respectively. The conditions tell us that the future-pointing outgoing null geodesics momentarily stop propagating outward.

Apparent horizons are defined quasi-locally and do not refer to the global causal structure of spacetime—they don't have the teleological nature of event horizons.

Apparent horizons are, in general, distinct from event horizons: for example, during the spherical collapse of uncharged matter, an event horizon forms before the apparent horizon does and these two horizons approach each other until they eventually coincide as the final static state is reached.



Killing horizons

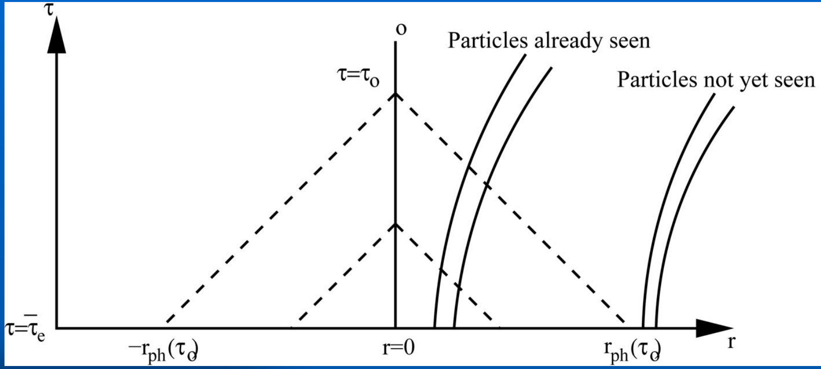
A Killing vector field k^a is one that satisfies the Killing equation

$$\nabla_a k_b + \nabla_b k_a = 0\,.$$

A Killing vector describes a symmetry of spacetime in a geometric, coordinate-invariant way.

A Killing horizon H of the spacetime (M, g_{ab}) is a null hyper-surface which is everywhere tangent to a Killing vector field k_a which becomes null, $k^c k_c = 0$, on H. This Killing vector field is time-like, $k^c k_c < 0$, in a spacetime region which has H as boundary. Stationary event horizons in General Relativity are usually Killing horizons for a suitably chosen Killing vector.

Cosmological horizon



If

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2_{(2)}\right)$$

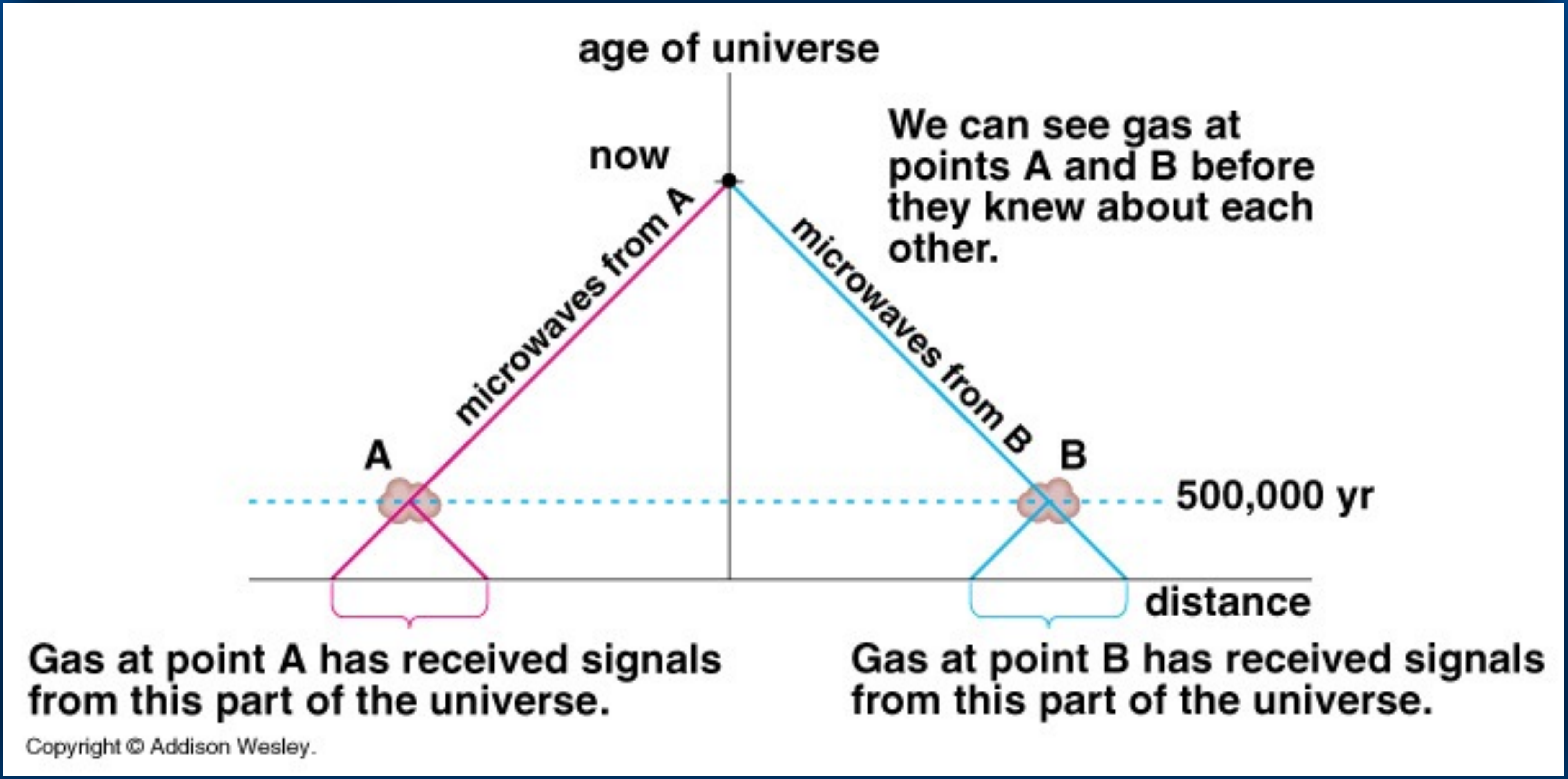
The cosmological or particle horizon at time t is a sphere centered on the co-moving observer at $r = 0$ and with radius:

$$\eta(t) = \int_0^t \frac{dt'}{a(t')}$$

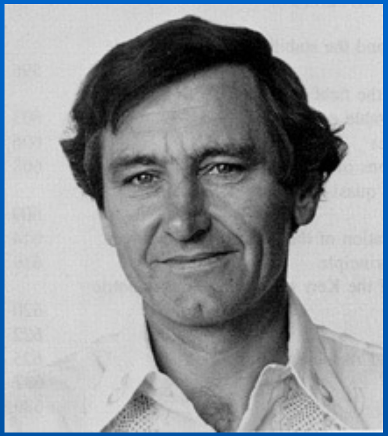
$$R_{\text{PH}}(t) = a(t)\int_0^t \frac{dt'}{a(t')}\,.$$

The particle horizon contains every particle signal that has reached the observer between the time of the Big Bang $t = 0$ and the time t .

Horizon problem



Kerr black holes



Roy Kerr

The Kerr metric corresponds to rotating body of mass M and angular momentum per unit mass a.

$$J = Ma.$$

$$\begin{aligned}
ds^2 &= g_{tt}dt^2 + 2g_{t\phi}dtd\phi - g_{\phi\phi}d\phi^2 - \Sigma\Delta^{-1}dr^2 - \Sigma d\theta^2 \\
g_{tt} &= (c^2 - 2GMr\Sigma^{-1}) \\
g_{t\phi} &= 2GMac^{-2}\Sigma^{-1}r\sin^2\theta \\
g_{\phi\phi} &= [(r^2 + a^2c^{-2})^2 - a^2c^{-2}\Delta\sin^2\theta]\Sigma^{-1}\sin^2\theta \\
\Sigma &\equiv r^2 + a^2c^{-2}\cos^2\theta \\
\Delta &\equiv r^2 - 2GMc^{-2}r + a^2c^{-2}.
\end{aligned}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{tt} & 0 & 0 & g_{t\phi} \\ 0 & \frac{\Sigma}{\Delta} & 0 & 0 \\ 0 & 0 & \Sigma & 0 \\ g_{t\phi} & 0 & 0 & g_{\phi\phi} \end{pmatrix}$$

The horizon, the surface which cannot be crossed outward, is determined by the condition $g_{rr} \to \infty$ $(\Delta = 0)$. It lies at $r = r_h$ where

$$r_h \equiv GMc^{-2} + [(GMc^{-2})^2 - a^2c^{-2}]^{1/2}.$$

There is also an inner horizon:

$$r_h^{\text{inn}} \equiv GMc^{-2} - [(GMc^{-2})^2 - a^2c^{-2}]^{1/2}.$$

limit $a \to 0$,

$$ds^2 \to c^2\left(1-\frac{2\mu}{r}\right)dt^2-\left(1-\frac{2\mu}{r}\right)^{-1}dr^2-r^2\,d\theta^2-r^2\sin^2\theta\,d\phi^2.$$

$\mu = GM/c^2$.

$$ds^2 = c^2\,dt^2-\frac{\rho^2}{r^2+a^2}\,dr^2-\rho^2\,d\theta^2-(r^2+a^2)\sin^2\theta\,d\phi^2.$$

$\mu \to 0$,

$$\rho^2 = r^2+a^2\cos^2\theta, \qquad \Delta = r^2-2\mu r+a^2.$$

This is indeed the Minkowski metric $ds^2 = c^2\,dt^2 - dx^2 - dy^2 - dz^2$, but written in terms of spatial coordinates (r, θ, ϕ) that are related to Cartesian coordinates by[2]

$$x = \sqrt{r^2 + a^2}\,\sin\theta\,\cos\phi,$$

$$y = \sqrt{r^2 + a^2}\,\sin\theta\,\sin\phi,$$

$$z = r\cos\theta,$$

where $r \geq 0$, $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$

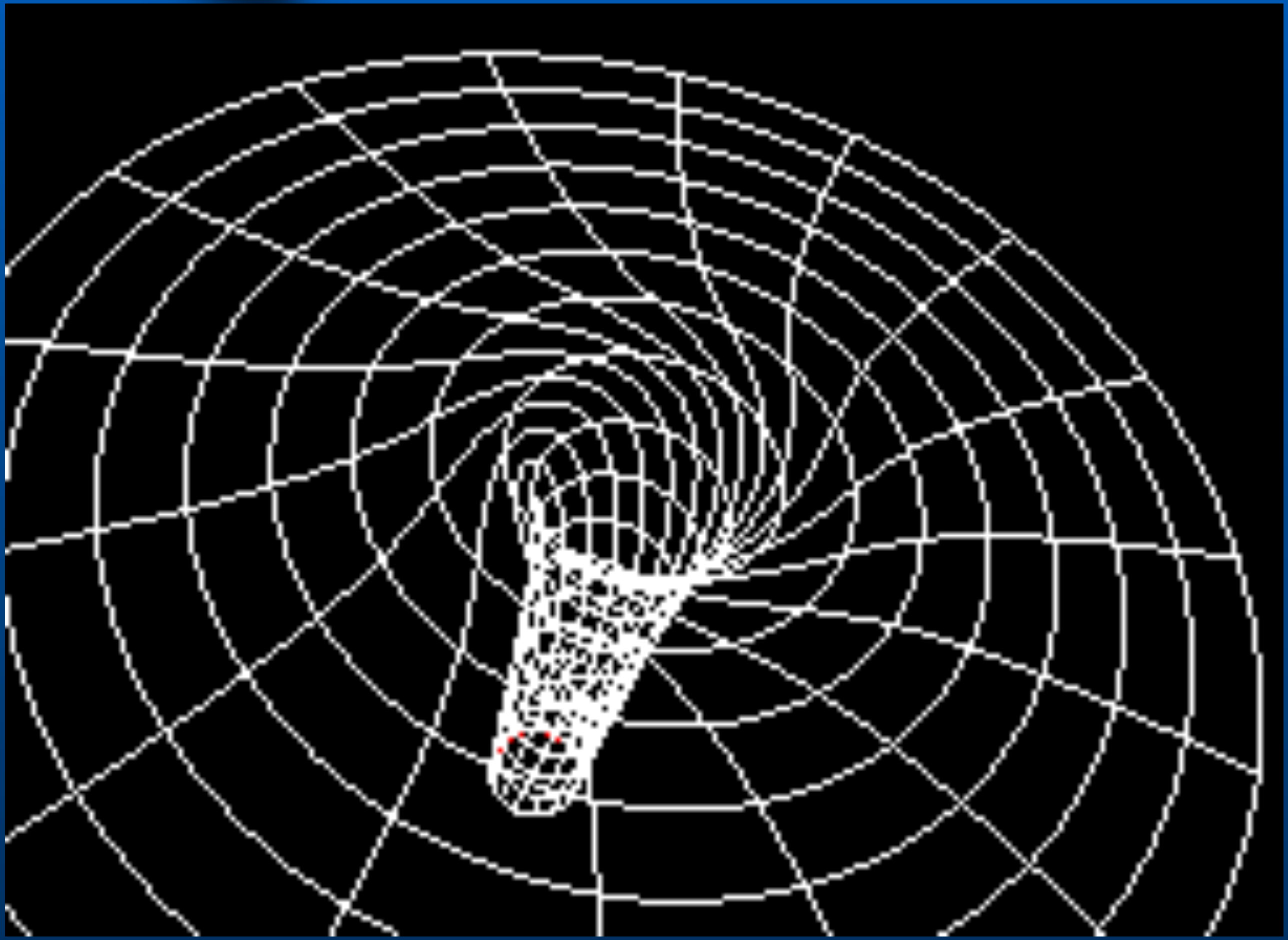
If $r \to +\infty$

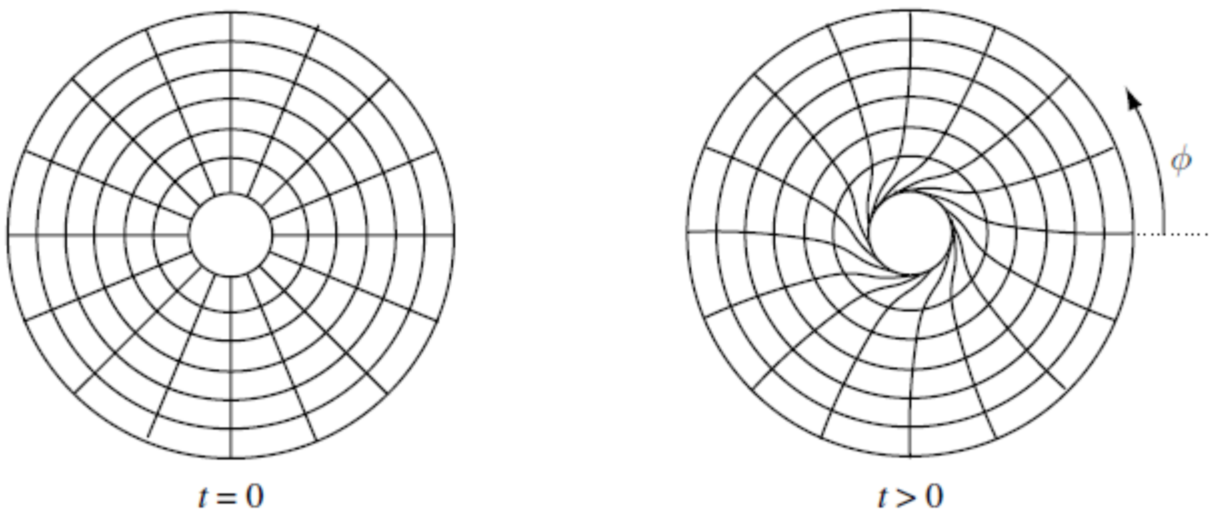
$$d\tau^2 = \left(1 - \frac{2m}{r}\right) dt^2 + \frac{4am\sin^2\theta}{r} dt d\phi - \left(1 + \frac{2m}{r}\right)^{-} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2.$$

Spacetime rotates away from the black hole.

This is called the “dragging of inertial frames”

Kerr space-time





A schematic illustration of the dragging of inertial frames around a rotating source.

Features of Kerr metric

- It is *stationary*: it does not depend explicitly on time.
- It is *axisymmetric*: it does not depend explicitly on ϕ.
- It is not static: it is not invariant for time reversal $t \to -t$.
- It is invariant for simultaneous inversion of t and ϕ,

$$\begin{aligned} t &\to -t \\ \phi &\to -\phi\,, \end{aligned}$$

  as can be expected: the time reversal of a rotating object produces an object which rotates in the opposite direction.
- In the limit $r \to \infty$, the Kerr metric reduces to Minkowski metric in polar coordinates; then, the Kerr spacetime is *asymptotically flat*.

The Kerr metric is the unique stationary axisymmetric vacuum solution (Carter-Robinson theorem).

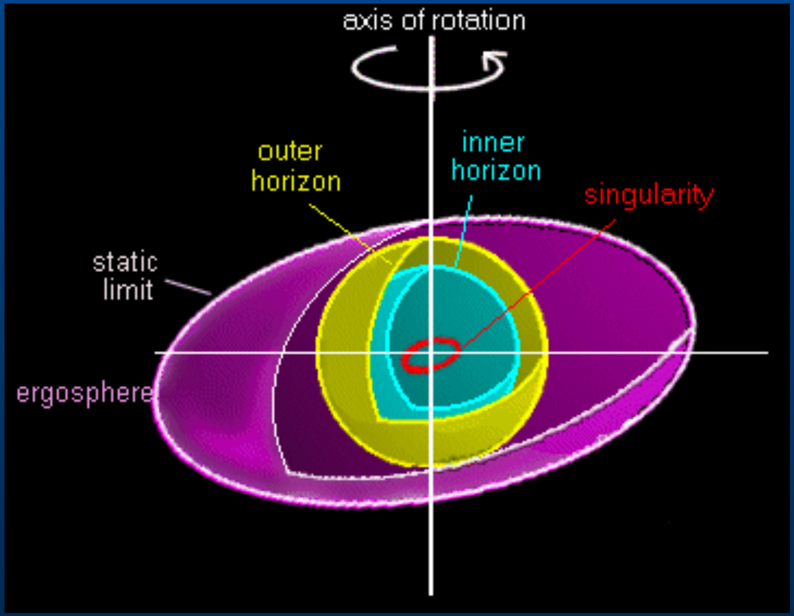
Horizons of the Kerr metric

The horizon, the surface which cannot be crossed outward, is determined by the condition $g_{rr} \to \infty$ $(\Delta = 0)$. It lies at $r = r_\mathrm{h}^\mathrm{out}$ where

$$r_\mathrm{h}^\mathrm{out} \equiv GMc^{-2} + \left[\left(GMc^{-2}\right)^2 - a^2c^{-2}\right]^{1/2}.$$

The second, the *inner* horizon, is located at:

$$r_\mathrm{h}^\mathrm{inn} \equiv GMc^{-2} - \left[\left(GMc^{-2}\right)^2 - a^2c^{-2}\right]^{1/2}.$$



The Kerr singularity

An essential singularity occurs when $g_{tt} \to \infty$

This happens in Kerr metric if $\Sigma = 0$

This condition implies:

$$r^2 + a^2 c^{-2} \cos^2\theta = 0.$$

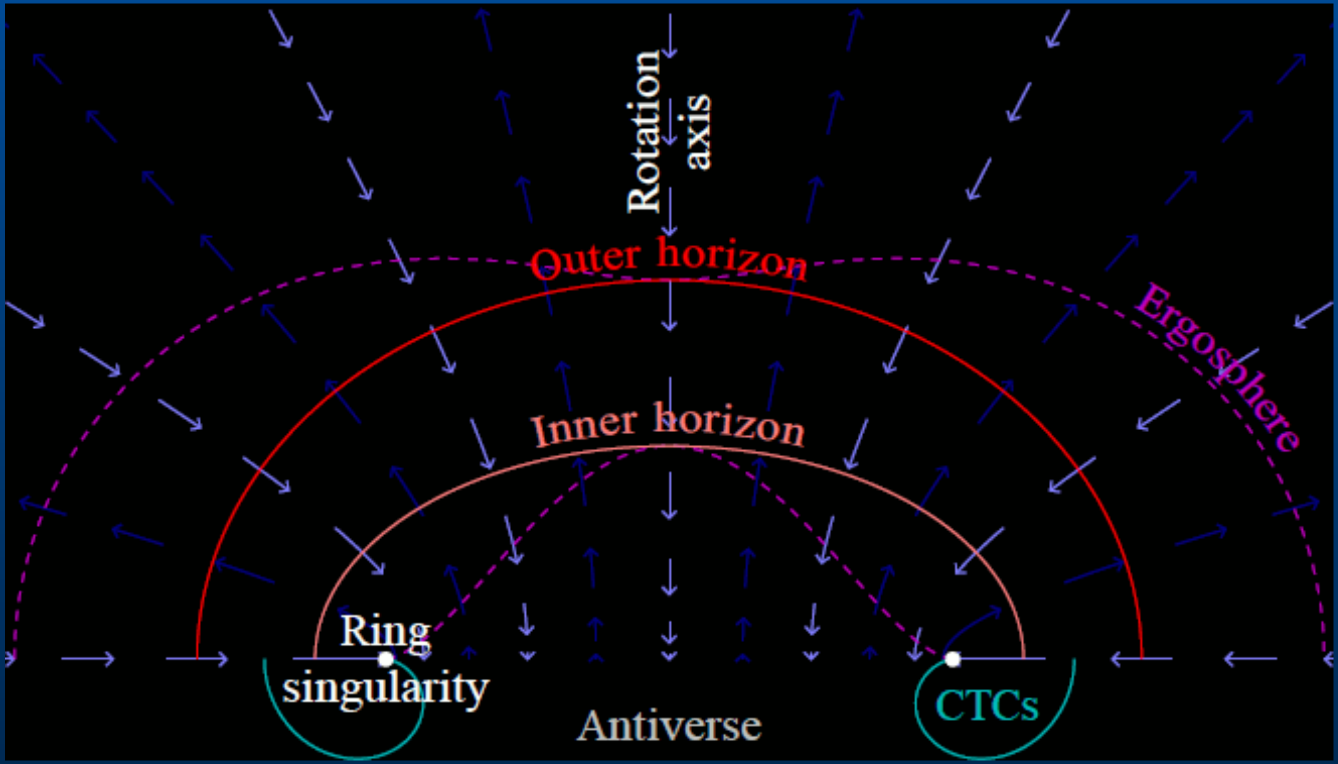
Such a condition is fulfilled only by $r = 0$ and $\theta = \frac{\pi}{2}$

This translates in Cartesian coordinates to:

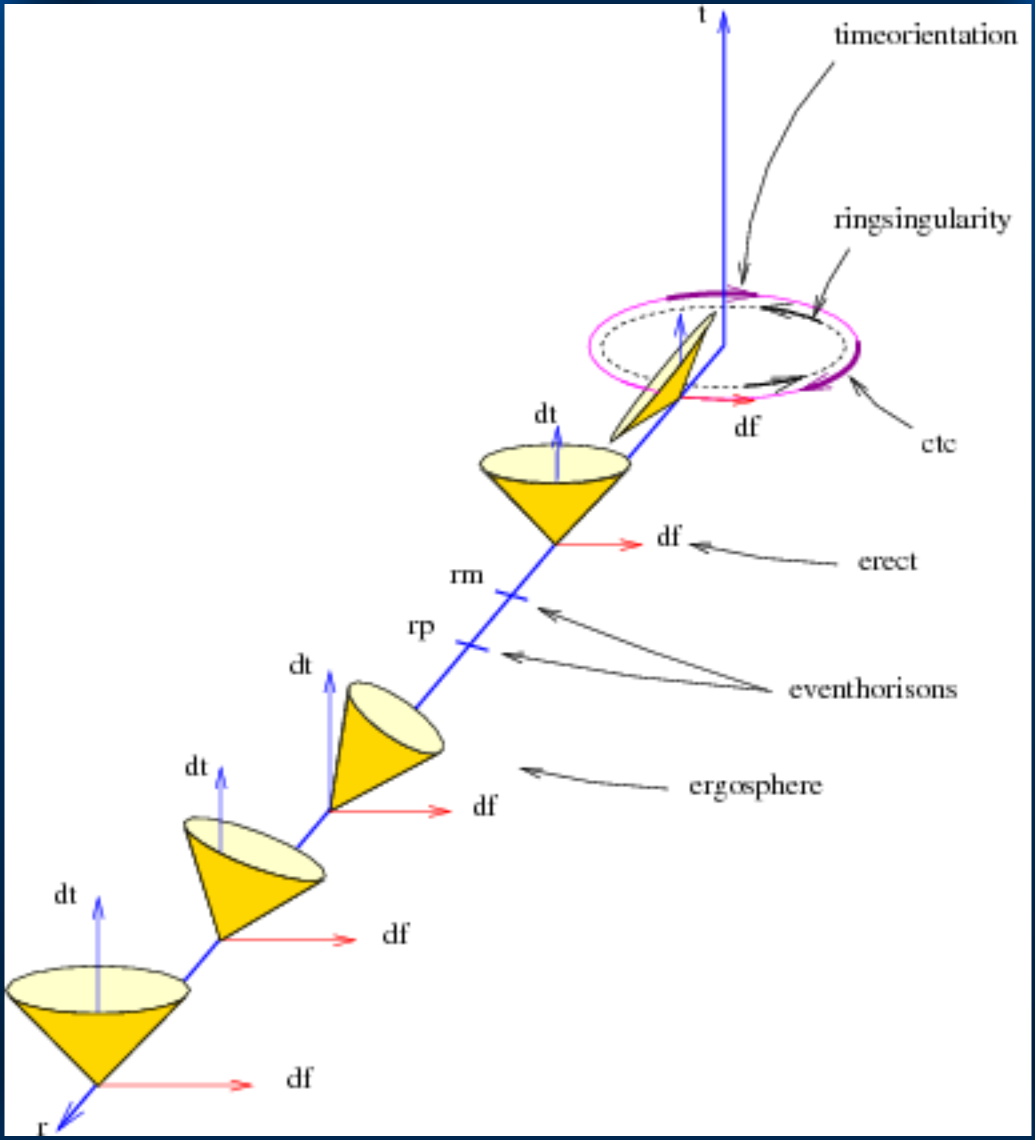
$$x^2 + y^2 = a^2 c^{-2} \quad \text{and} \quad z = 0.$$

The Kerr singularity

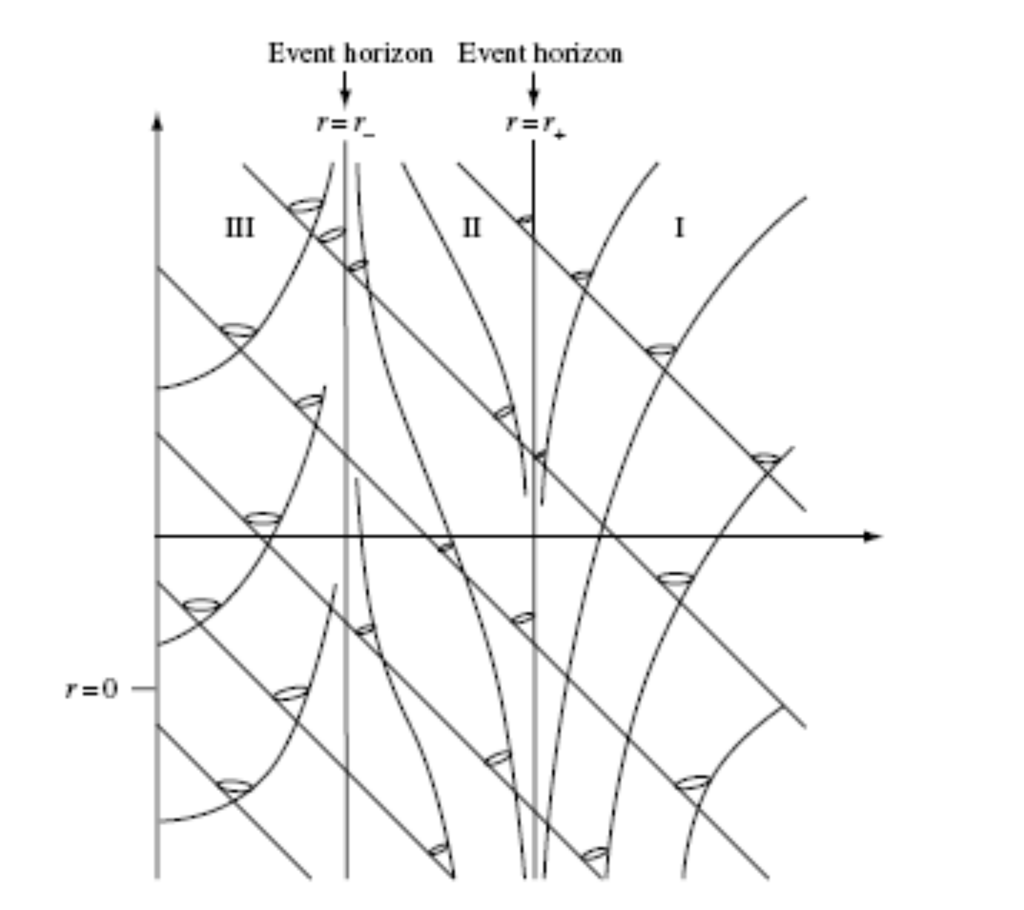
The singularity is a ring of radius ac^{-1} on the equatorial plane. If $a = 0$, then Schwarzschild's point-like singularity is recovered. If $a \neq 0$ the singularity is not necessarily in the future of all events at $r < r_{\mathrm{h}}^{\mathrm{inn}}$: the singularity can be avoided by some geodesics.



The Kerr singularity



Kerr space-time

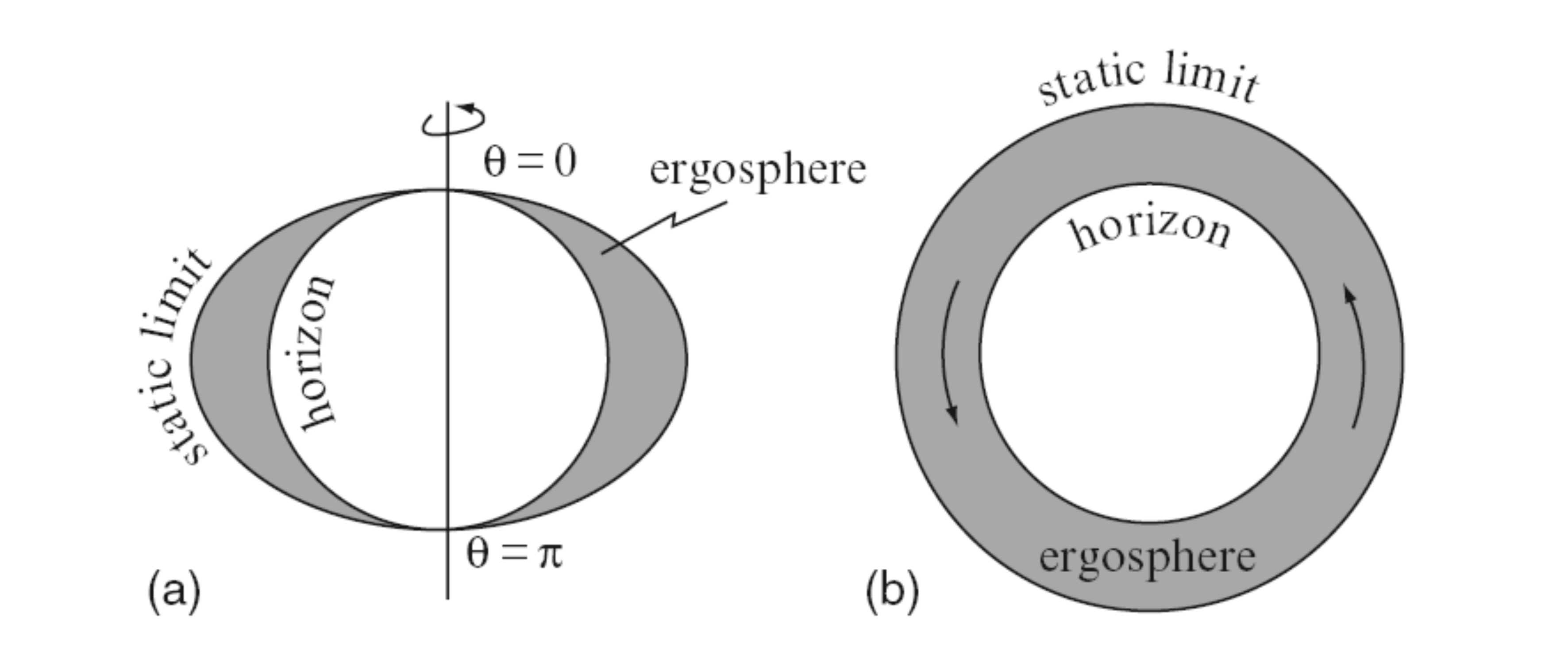


Ergosphere

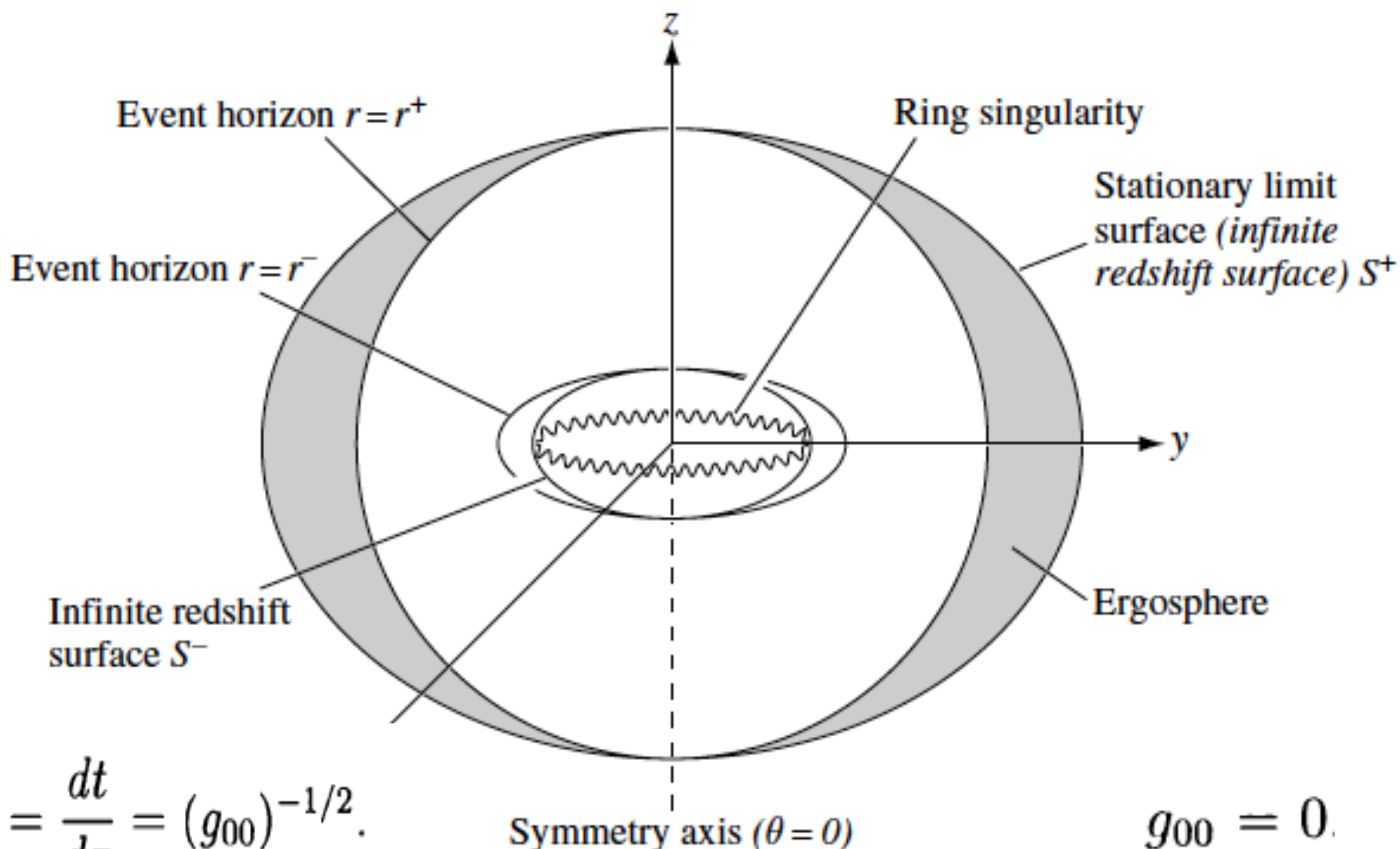
$$\boxed{g_{tt} = 0}$$

$$r_e = \frac{GM}{c^2} + \frac{1}{c^2}\left(G^2M^2 - a^2c^2\cos^2\theta\right)^{1/2}.$$

$$r_{S\pm} = \mu \pm \left(\mu^2 - a^2\cos^2\theta\right)^{1/2}.$$



Ergosphere



$$1 + z = \frac{dt}{d\tau} = (g_{00})^{-1/2}.$$

$$g_{00} = 0.$$

Ergosphere

If a particle initially falls radially with no angular momentum from infinity to the black hole, it gains angular motion during the infall. The angular velocity as seen from a distant observer is:

$$\Omega(r,\theta) = \frac{d\phi}{dt} = \frac{(2GM/c^2)ar}{(r^2 + a^2c^{-2})^2 - a^2c^{-2}\Delta\sin^2\theta}.$$

The ergosphere

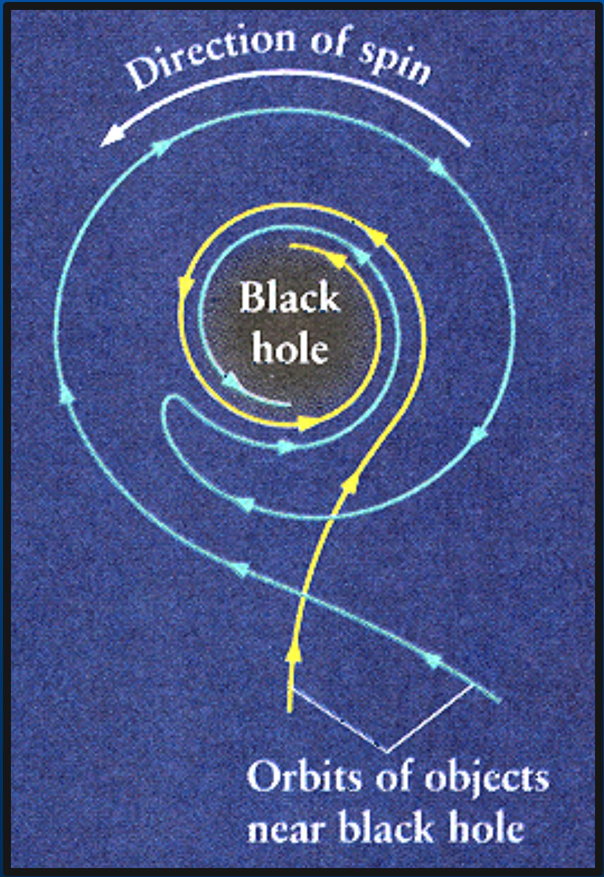
- Region between horizon and static limit
- **Nothing** can remain stationary in the ergosphere

Must rotate in direction of BH spin because
BH spin "drags space" along with it

aka:

"dragging of inertial frames"

"Lense-Thirring effect"



Why “ERGOSPHERE”?

- “ERGO” = ENERGY
- All the spin energy of a black hole resides outside the horizon!! it can *all* be extracted (… in theory)
- For maximal Kerr hole with mass *M*:

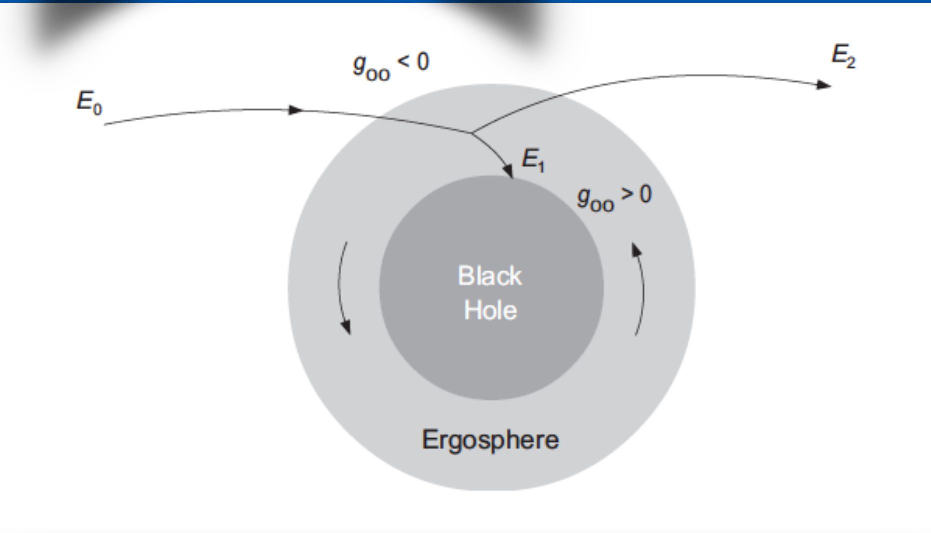
SPIN ENERGY = 29% of Mc^2

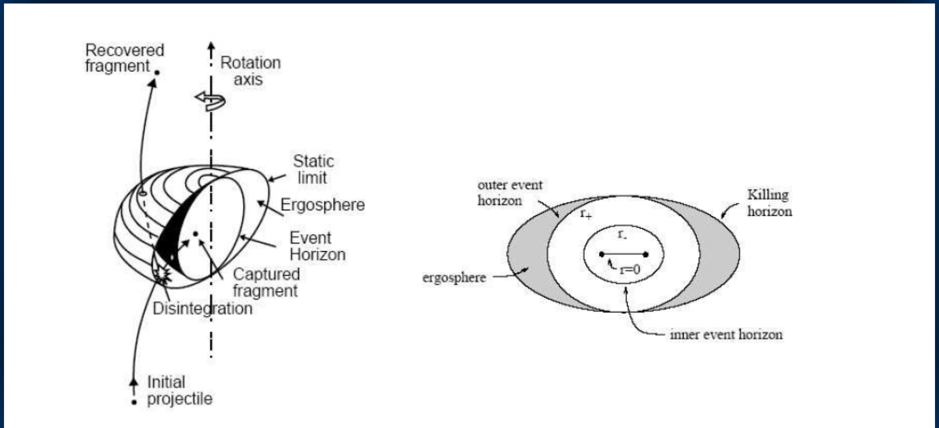
Two famous energy extraction schemes:

Penrose Process: particle splitting inside the ergosphere

Blandford-Znajek Process: BH spin twists magnetic field

Penrose process





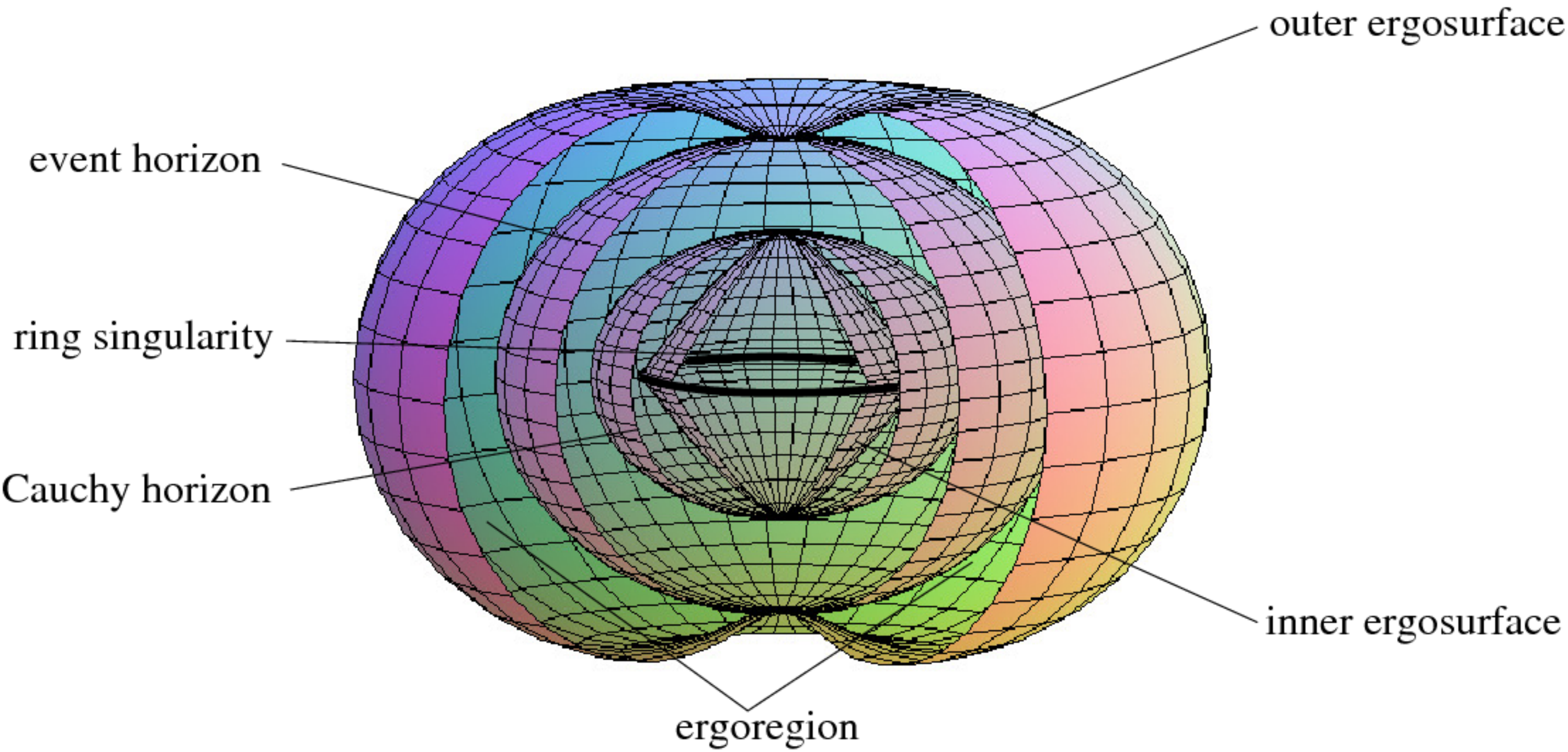
A particle following a geodesic that enters the ergosphere under some specific circumstances can decay into two particles A and B inside the ergosphere. The ingoing particle has an energy E which is equal to p_0 at infinity.

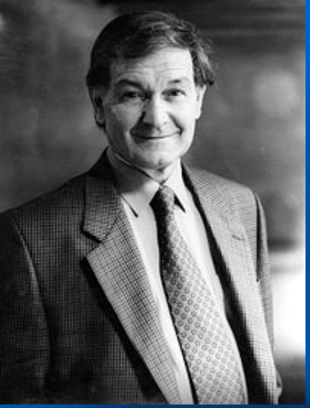
This particle decays into two particles A and B, with energies E_A and E_B: $E = E_A + E_B$. The decay can be done in such a way that particle B goes through the event horizon into the black hole, and particle A escapes from the black hole to infinity. Because of (global) energy conservation:

$$E_{\text{black hole; initial}} + E = E_{\text{black hole; final}} + E_A$$

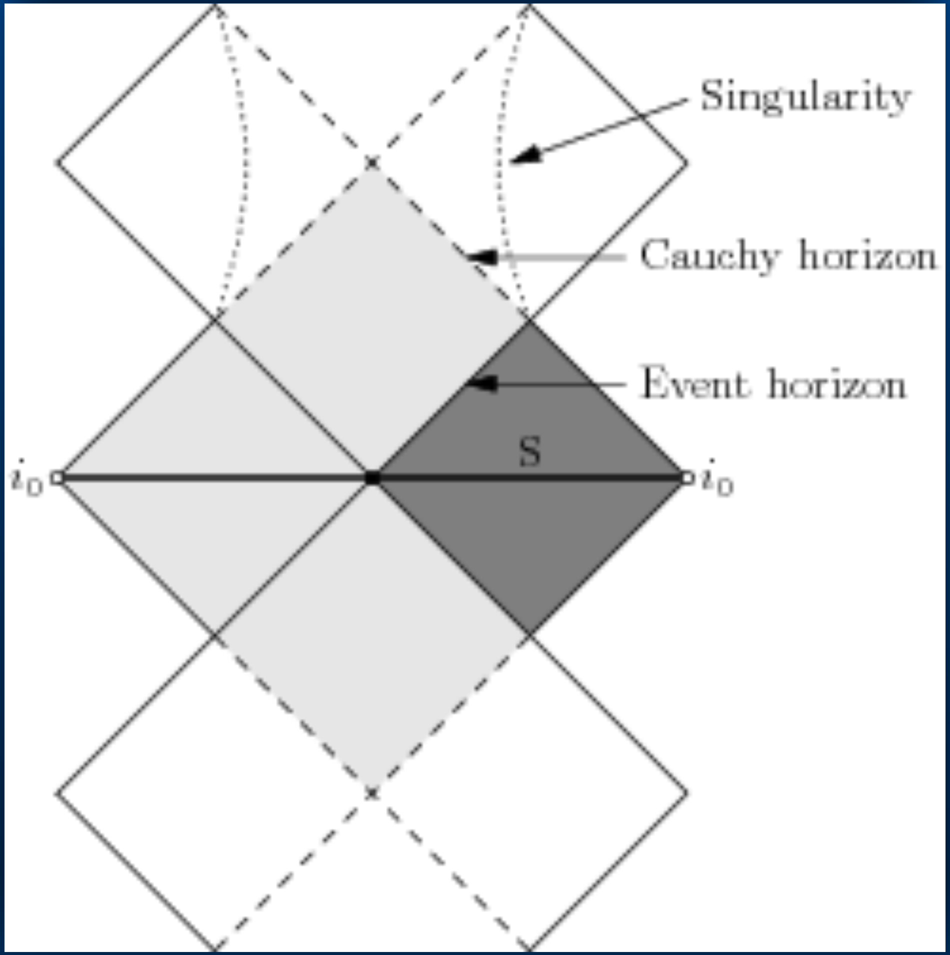
Particle B, crossing the event horizon, has a negative energy because within the ergosphere, the sign of the killing vector *t* changes. The black hole absorbs a negative energy. Particle A, that goes to infinity will gain that amount of energy because of energy conservation: $E_A > E$.

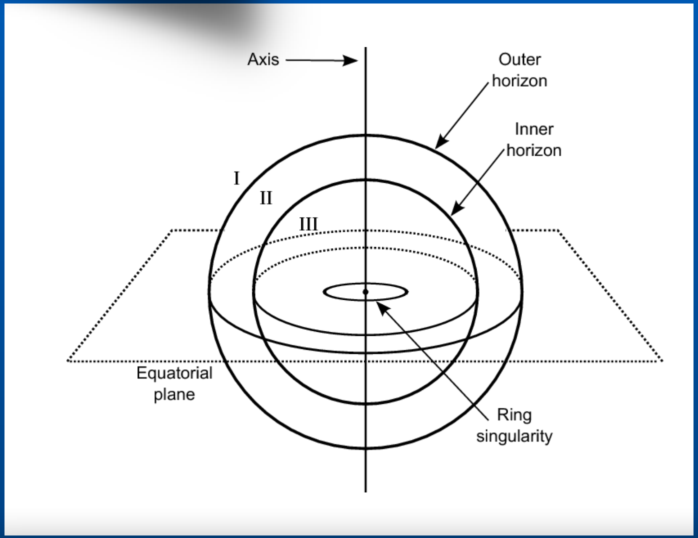
a<m

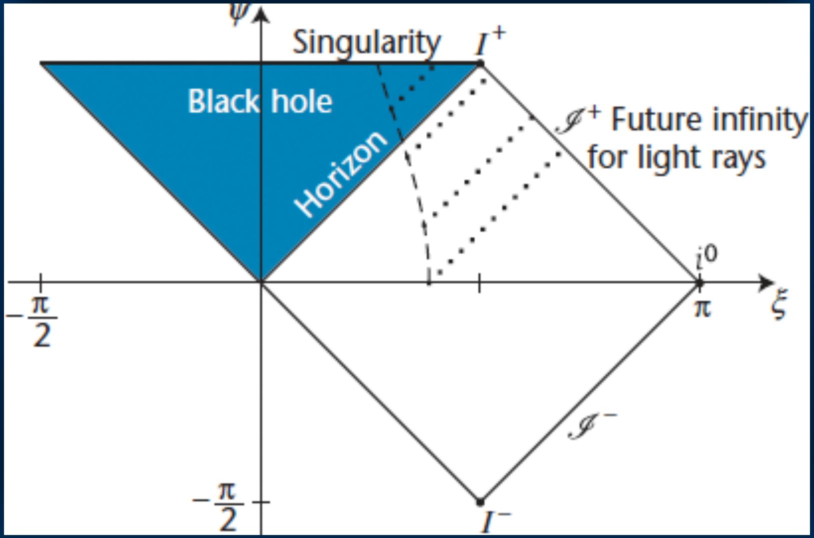

outer ergosurface
event horizon
ring singularity
Cauchy horizon
inner ergosurface
ergoregion



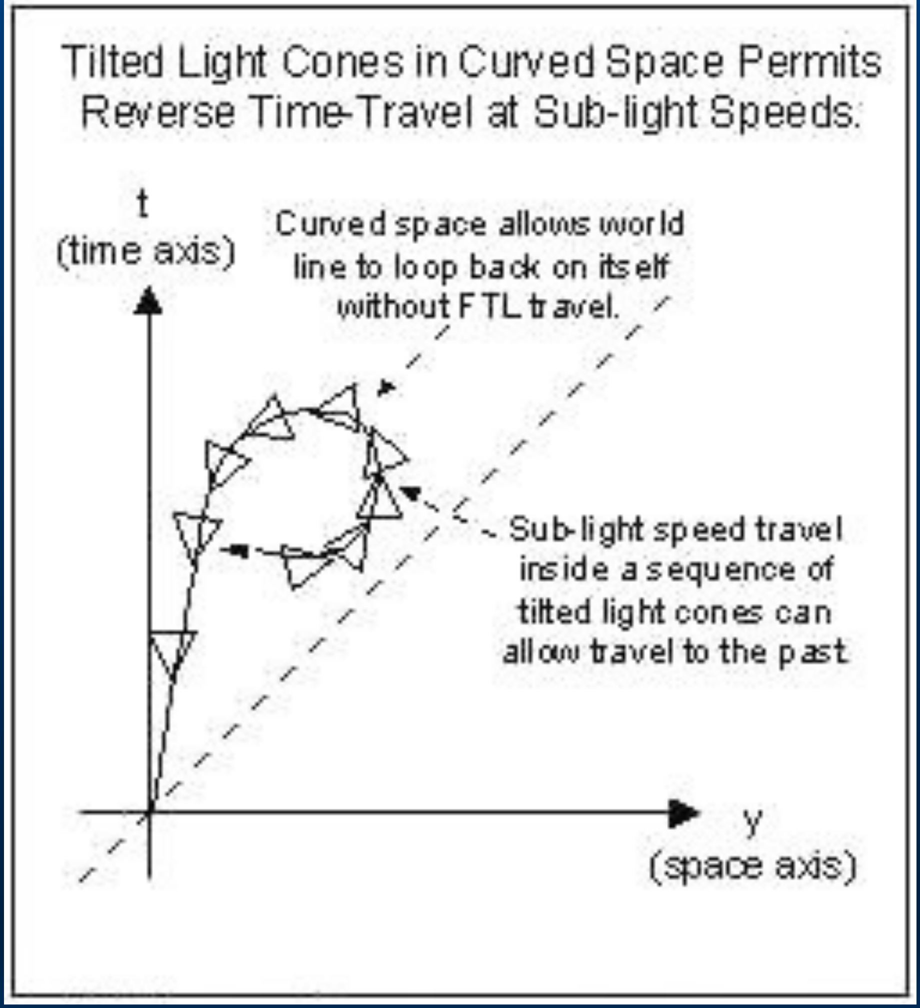
Penrose diagram for Kerr black holes



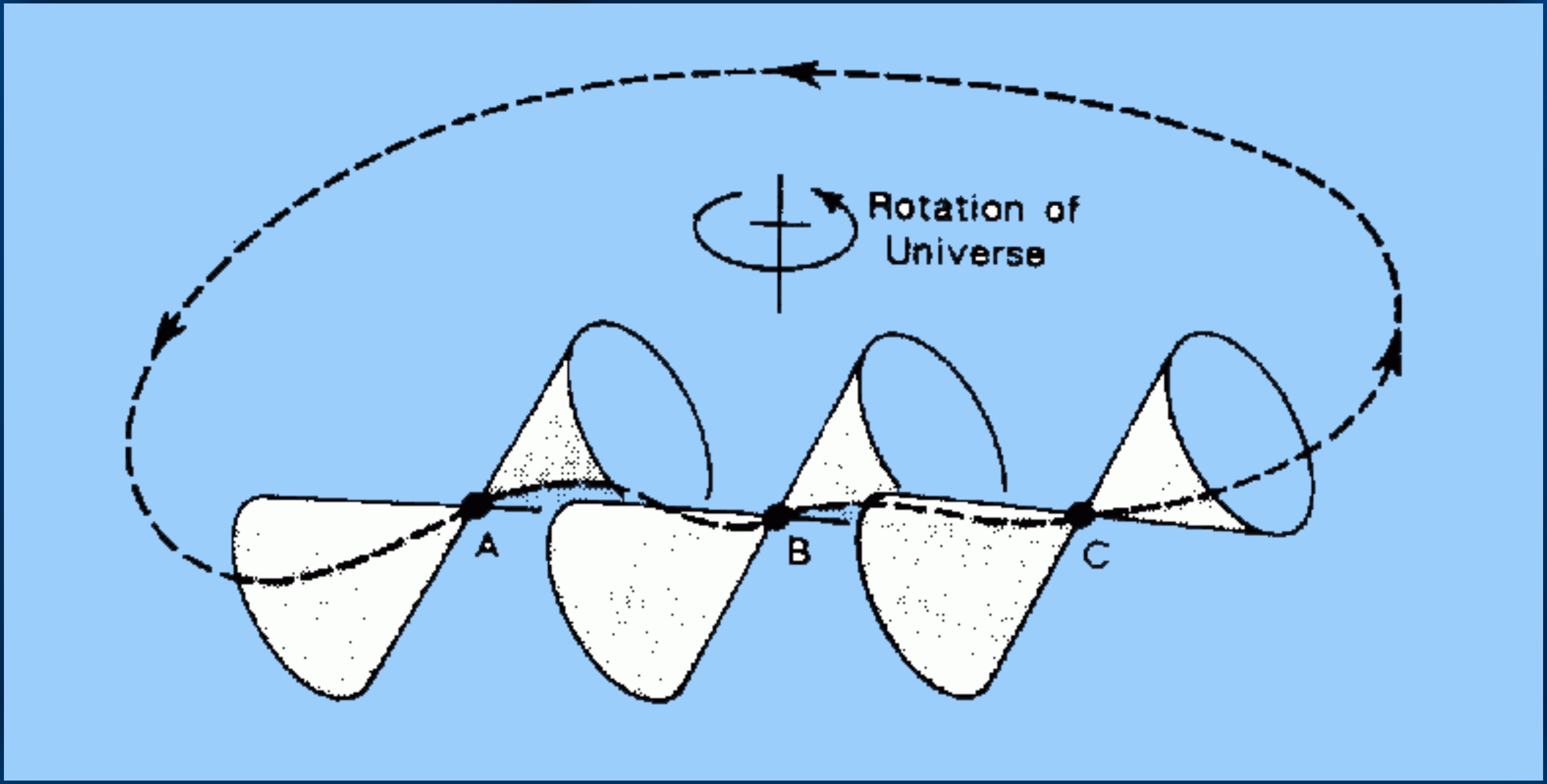




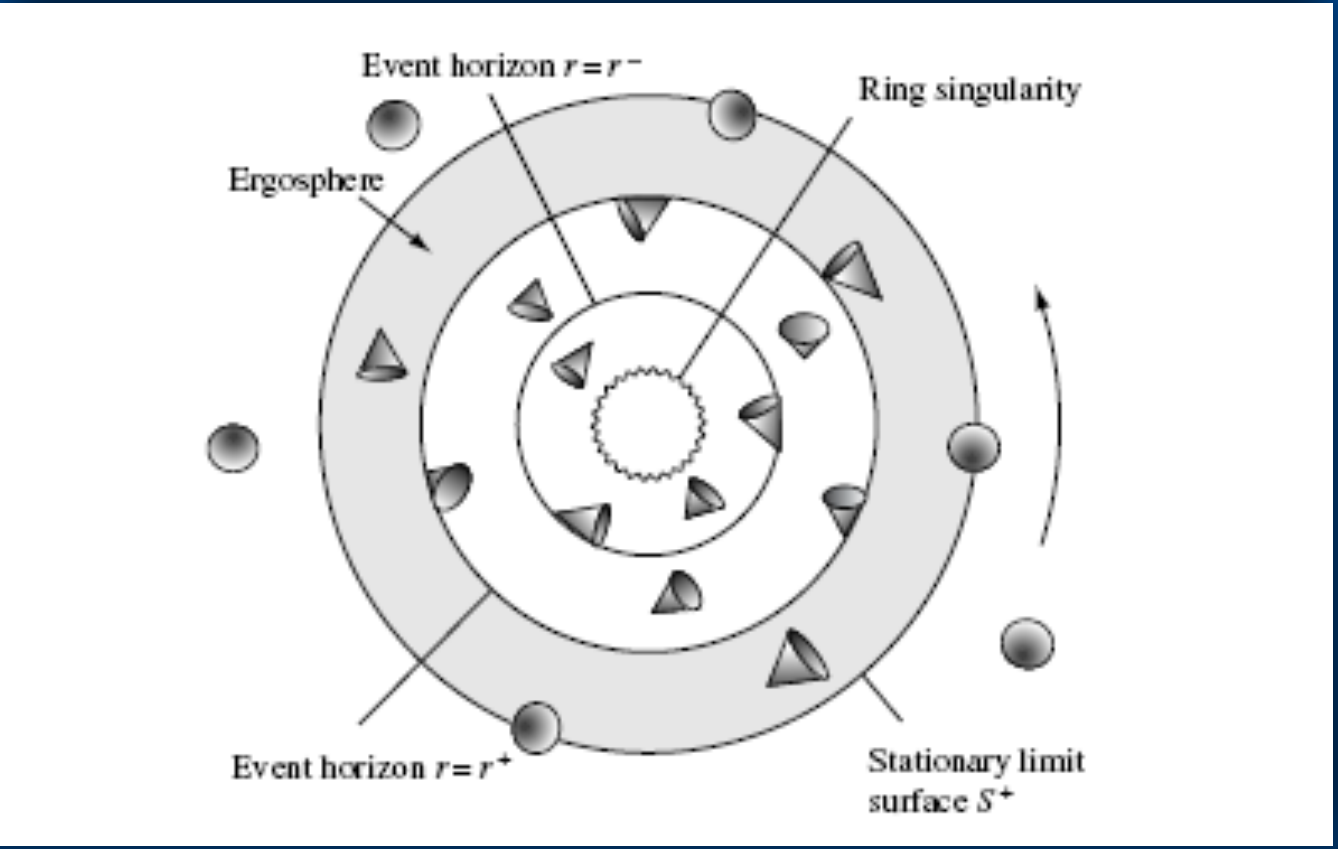
Closed time-like curves



Closed time-like curves

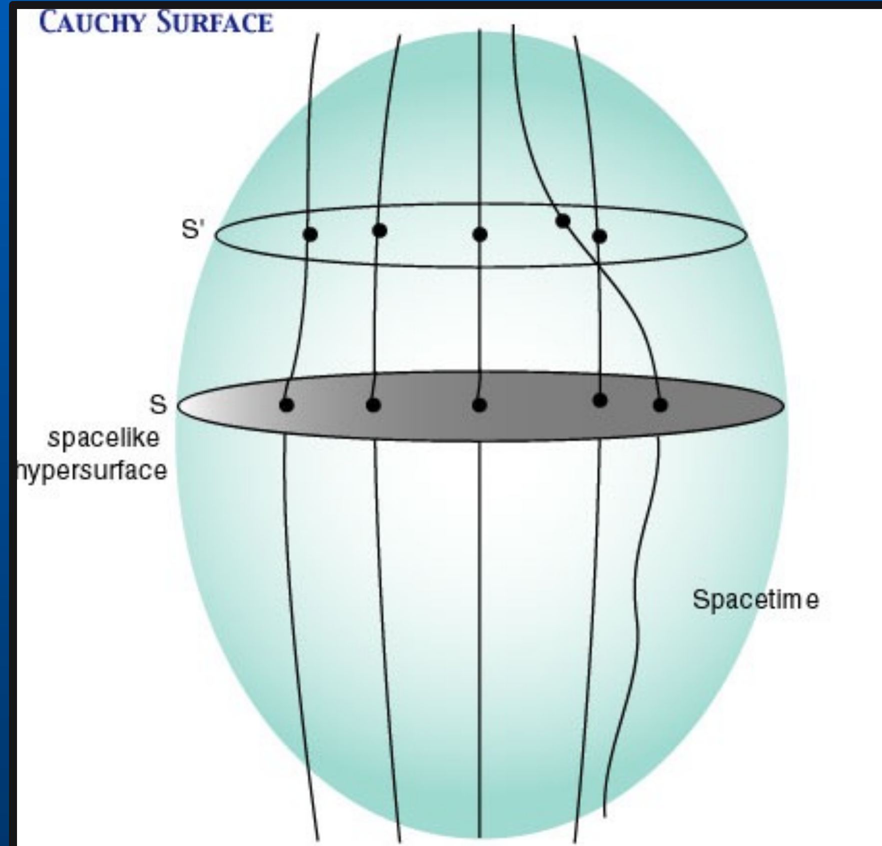


Kerr space-time and CTCs



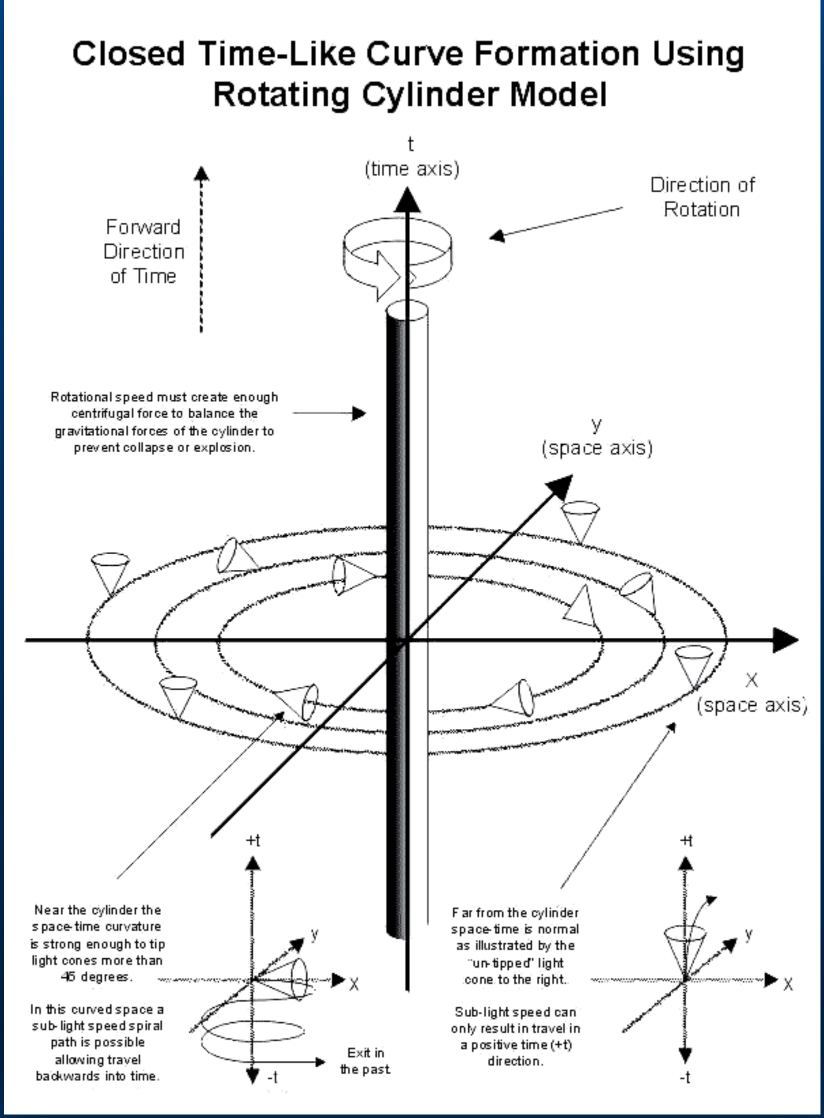
Cauchy horizons

If CTCs are formed in the immediate future of a surface S, then S becomes a Cauchy horizon and global determinism fails (predictability on the basis of GR).

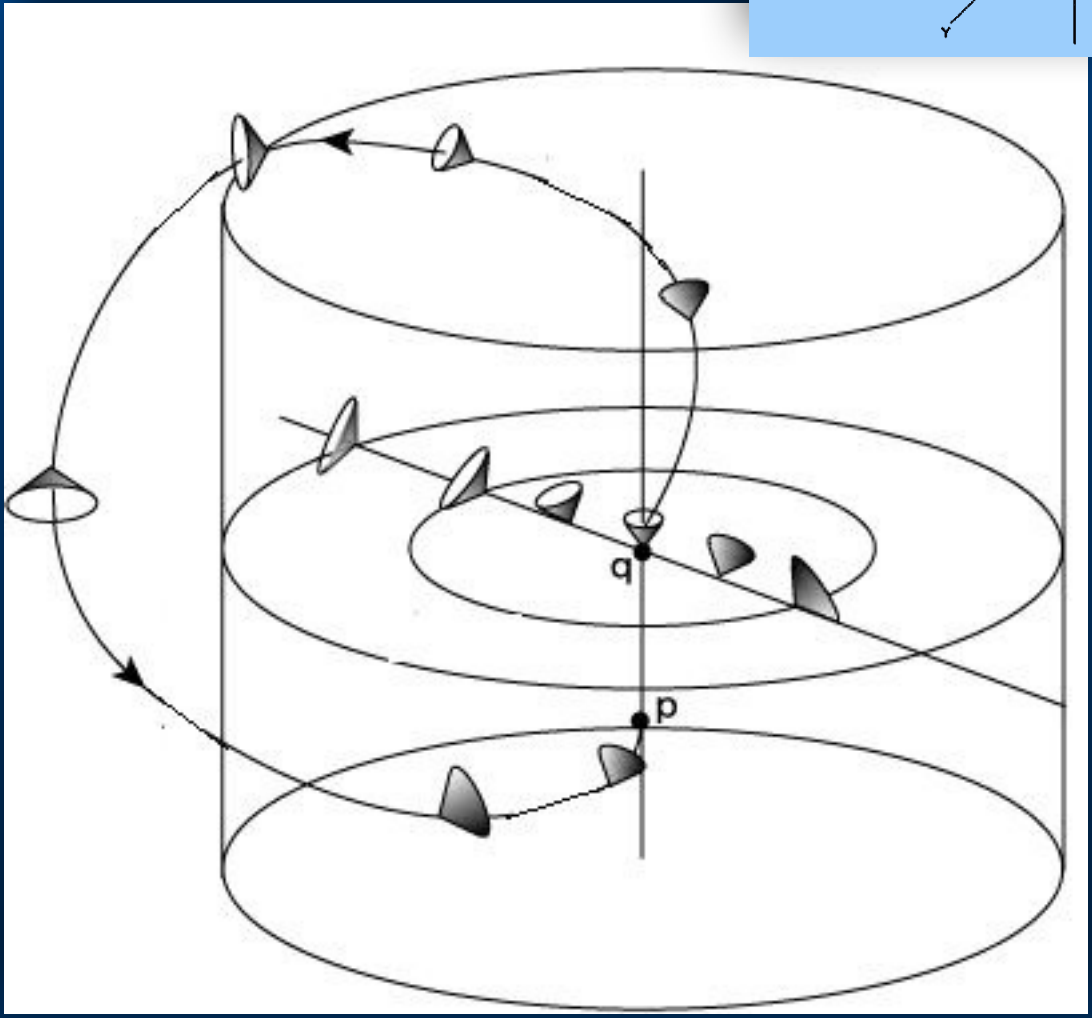


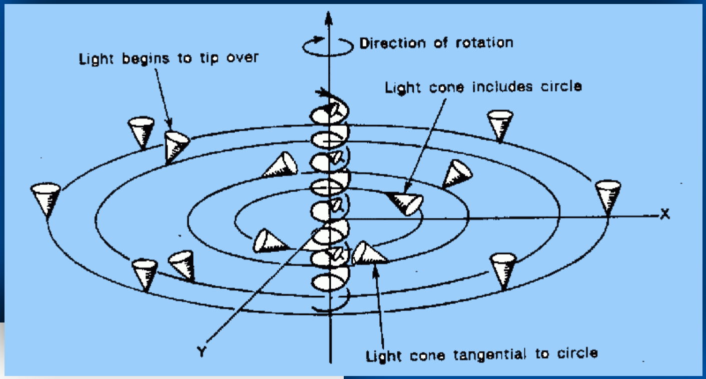
S is called **Cauchy surface** if ***every*** world-line (timelike curve) without endpoint intersects once and only once the hypersurface S.
If S and S' are a Cauchy surface, the events on S' are determined by those on S, if the law governing this spacetime is deterministic.
General relativity allows non-existence of Cauchy surfaces in certain cases.

Rotating massive cylinders



Gödel universe

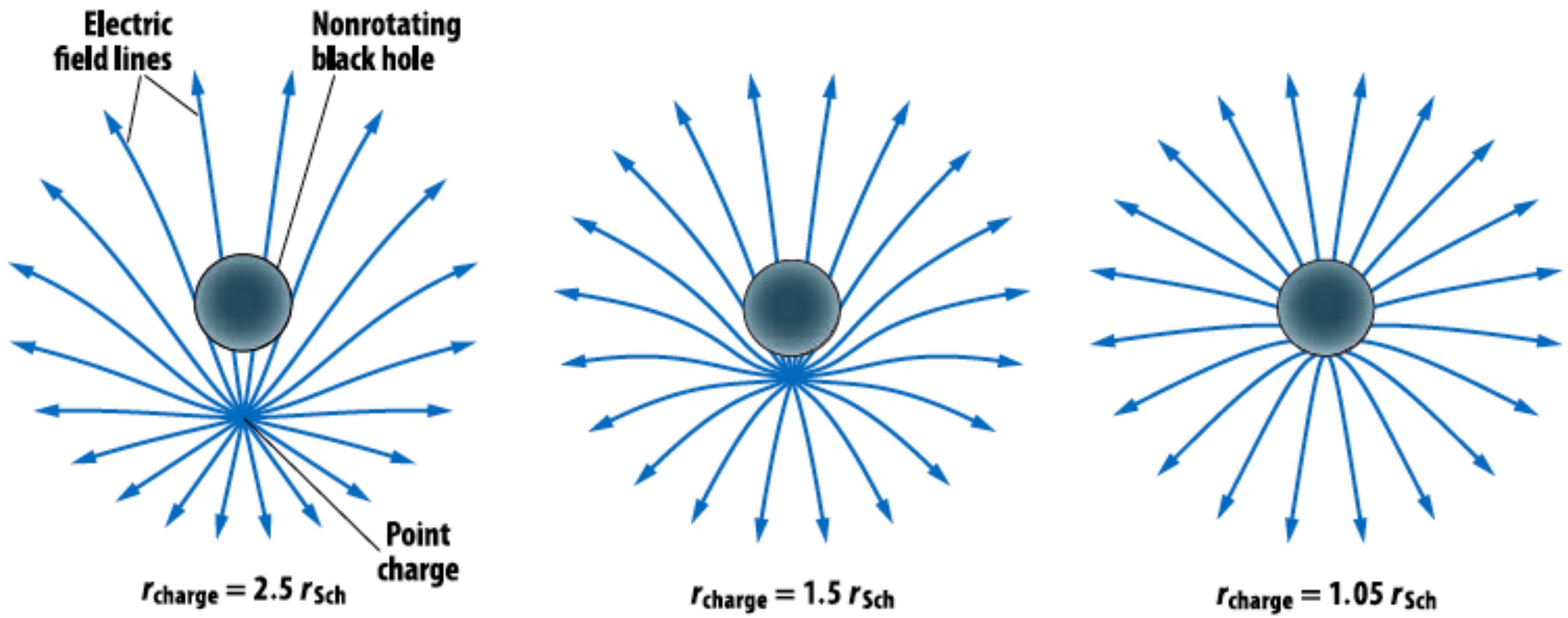




Tipler cylinder

Reissner-Nordström black holes

The Reissner-Nordström metric is a spherically symmetric solution of Einstein field equations. However, it is not a vacuum solution, since the source has an electric charge Q, and hence there is an electromagnetic field.



Reissner-Nordström black holes

The solution for the metric is given by

$$ds^2 = \Delta c^2 dt^2 - \Delta^{-1} dr^2 - r^2 d\Omega^2 \ , \tag{195}$$

where

$$\Delta = 1 - \frac{2GM/c^2}{r} + \frac{q^2}{r^2}. \tag{196}$$

In this expression, M is once again interpreted as the mass of the hole and

$$q = \frac{GQ^2}{4\pi\epsilon_0 c^4} \tag{197}$$

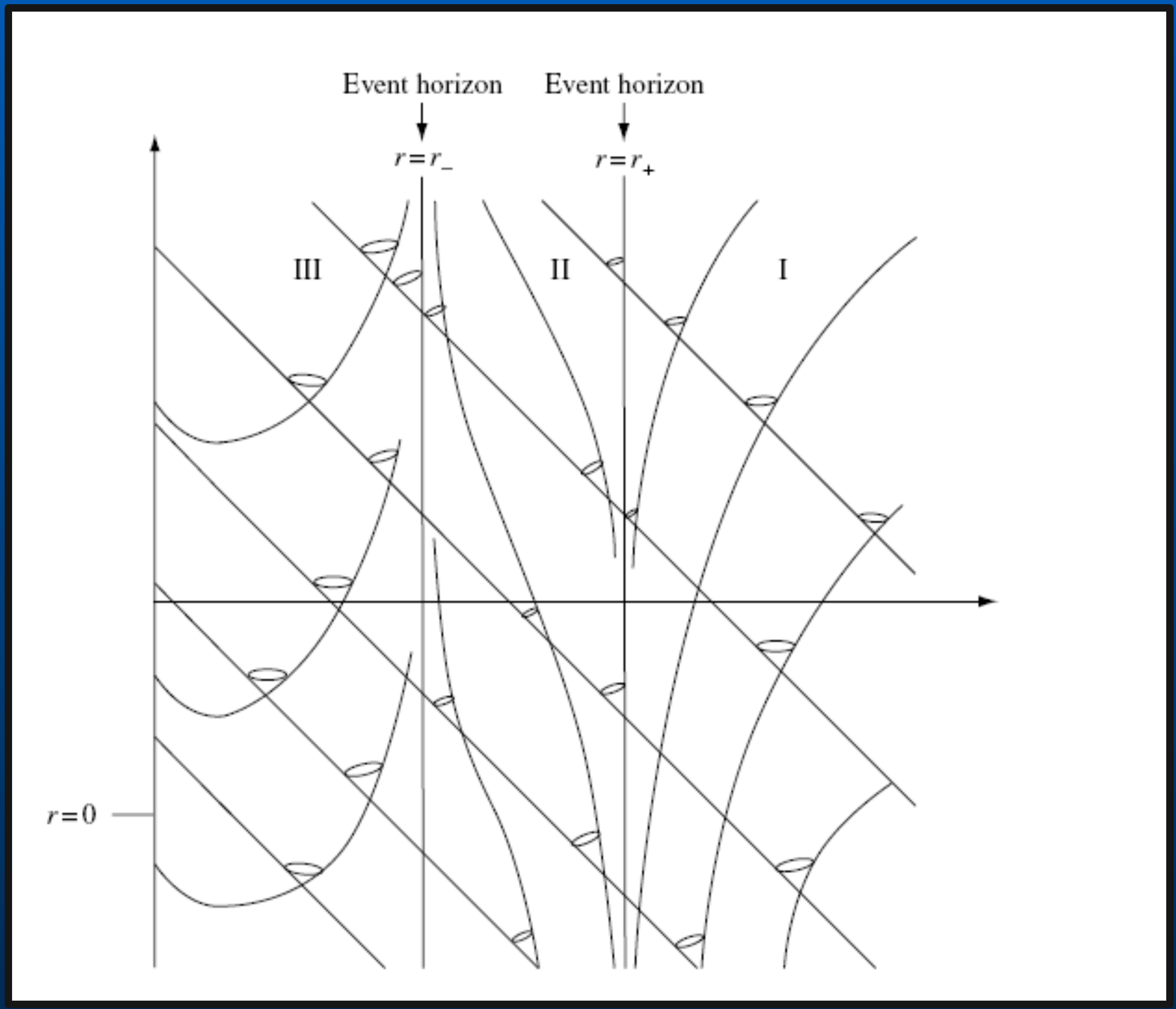
is related to the total electric charge Q.

The metric has a coordinate singularity at $\Delta = 0$, in such a way that:

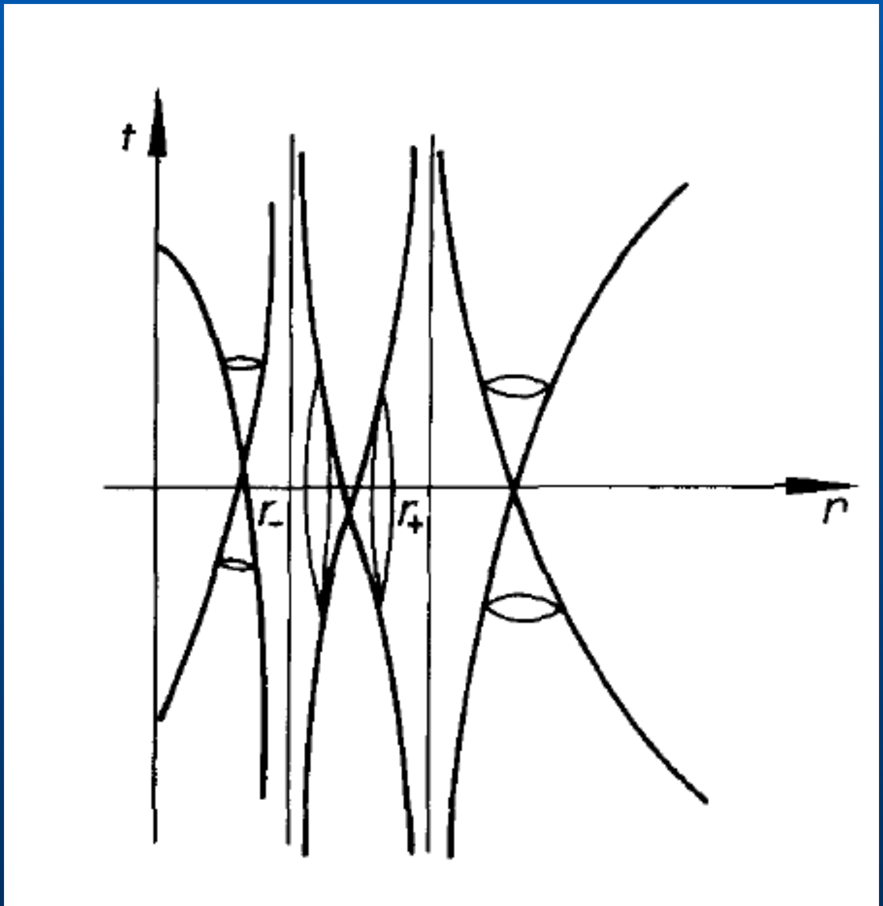
$$r_\pm = r_\mathrm{g} \pm (r_\mathrm{g}^2 - q^2)^{1/2}.$$

- For $r_g=q$, we have an extreme Reissner-Nordström black hole with a unique horizon at $r=r_g$.
- A Reissner-Nordström black hole can be more compact than a Scharzschild black hole of the same mass.
- For the case $r_g^2>q^2$, both $r+$ and $r-$ are real and there are two horizons as in the Kerr solution.
- In the case $r_g^2<q^2$ both $r+$ and $r-$ are imaginary and there is no coordinate singularities, no horizon hides the intrinsic singularity at $r=0$.

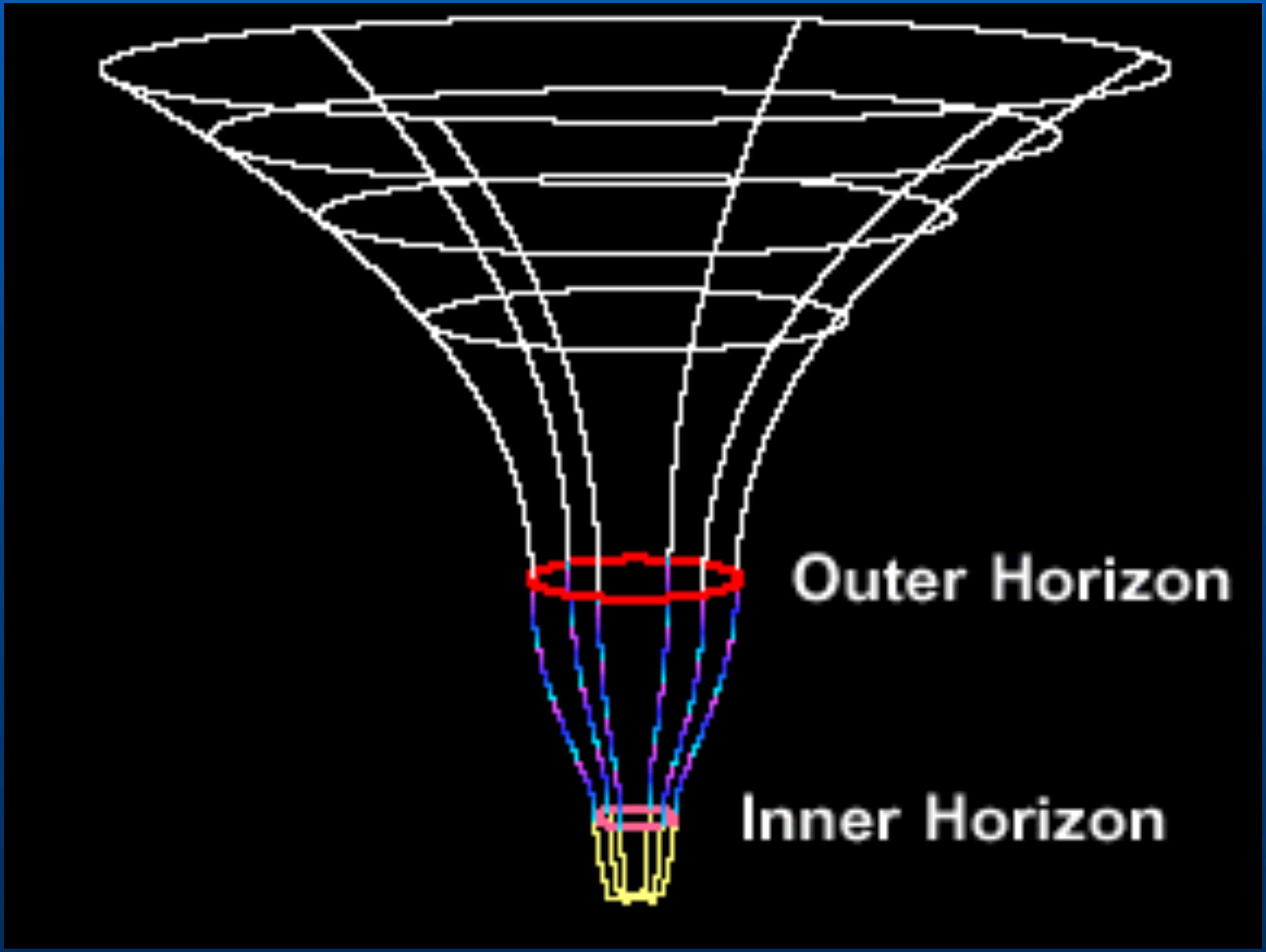
Reisner-Nordström black holes



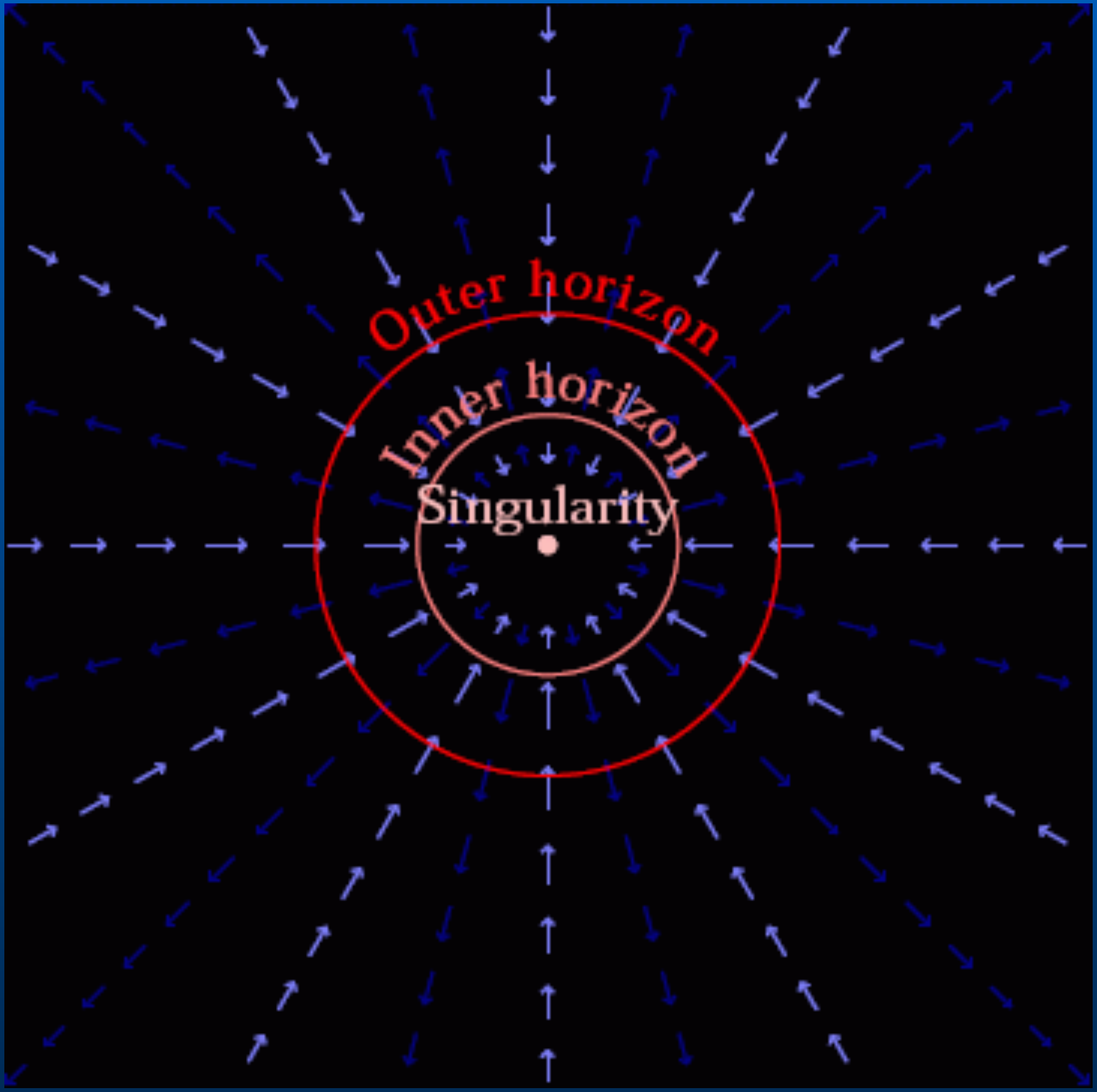
Reisner-Nordström black holes



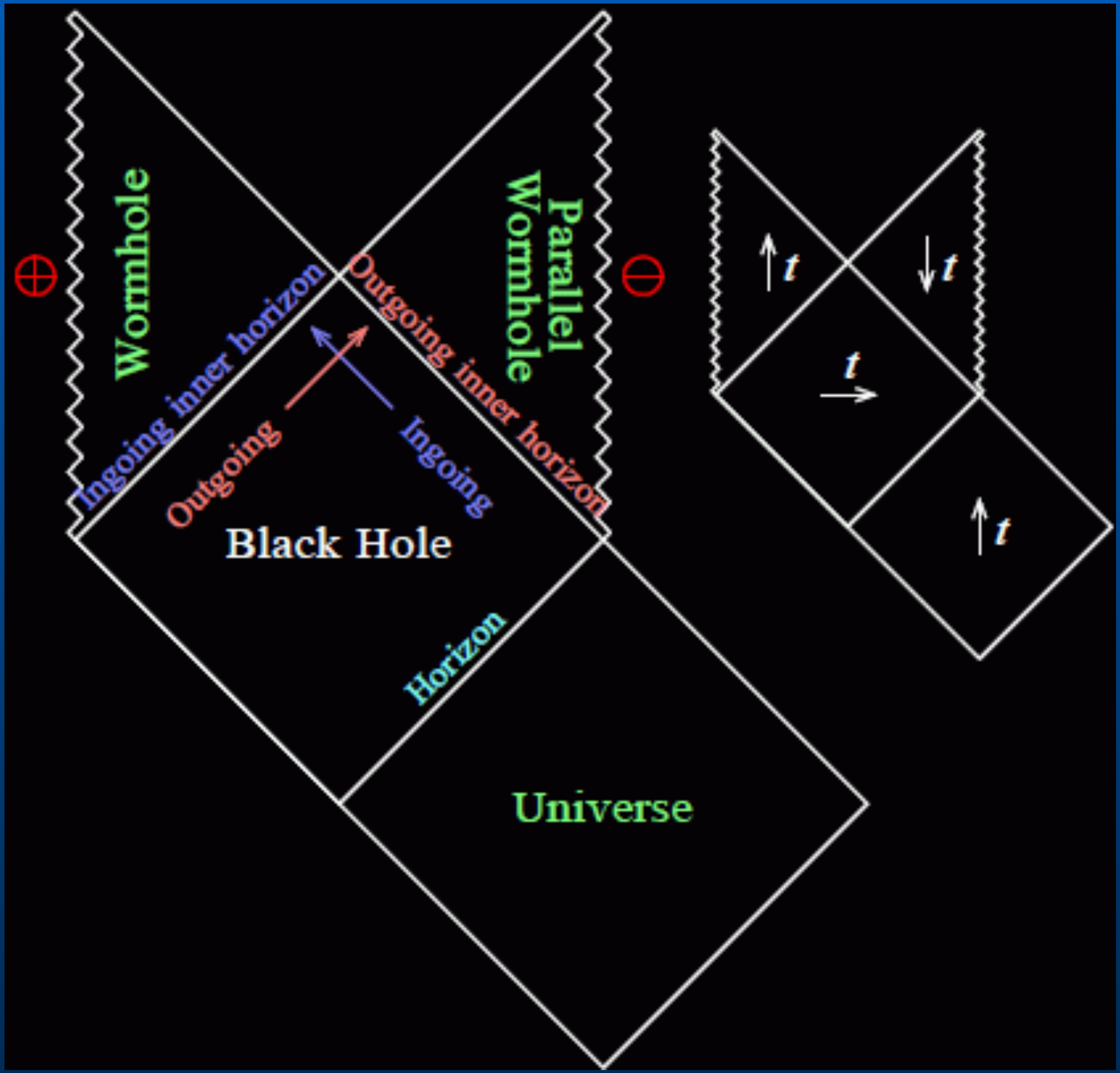
Reisner-Nordström black holes



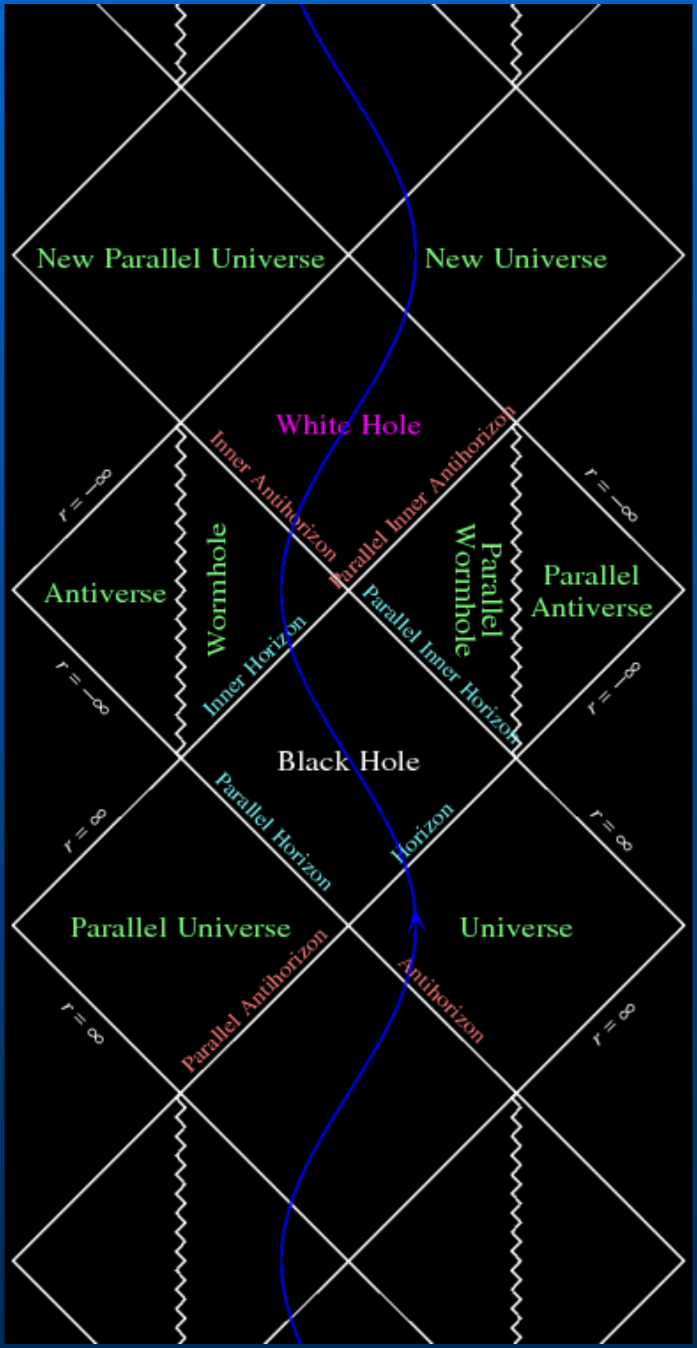
Reisner-Nordström black holes



Reisner-Nordström black holes



Reisner-Nordström black holes



Reisner-Nordström singularity

$$\frac{dt}{d\tau} = \frac{r}{\Delta}(\gamma r - eQ)$$
$$\frac{dr}{d\tau} = -\frac{1}{r}\left[(\gamma r - eQ)^2 - \Delta\right]^{\frac{1}{2}}$$

$$\frac{d^2 r}{d\tau^2} = \frac{\beta\gamma - m}{r^2} + \frac{Q^2 - \beta^2}{r^3};$$

ß is the charge of the falling particle
γ Is the Lorentz factor

$$\frac{d^2r}{d\tau^2} = -\frac{1}{r^2}\left(m - \frac{Q^2}{r}\right)$$

Neutral particle

The particle feels a gravitational field of variable effective mass $m' = m - Q^2/r$

The effective mass becomes negative for $r < Q^2/m$

and repulsion makes the singularity time-like (avoidable).

Kerr-Newman black holes

The Kerr-Newman metric of a charged spinning black hole is the most general black hole solution.

$$\frac{2GM}{c^2}r \longrightarrow \frac{2GM}{c^2}r - q^2,$$

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}\,dt\,d\phi - g_{\phi\phi}\,d\phi^2 - \Sigma\Delta^{-1}dr^2 - \Sigma d\theta^2$$

$$g_{tt} = c^2\left[1 - (2GMrc^{-2} - q^2)\Sigma^{-1}\right]$$

$$g_{t\phi} = a\sin^2\theta\,\Sigma^{-1}\left(2GMrc^{-2} - q^2\right)$$

$$g_{\phi\phi} = [(r^2 + a^2c^{-2})^2 - a^2c^{-2}\Delta\sin^2\theta]\Sigma^{-1}\sin^2\theta$$

$$\Sigma \equiv r^2 + a^2c^{-2}\cos^2\theta$$

$$\Delta \equiv r^2 - 2GMc^{-2}r + a^2c^{-2} + q^2 \equiv (r - r_{\mathrm{h}}^{\mathrm{out}})(r - r_{\mathrm{h}}^{\mathrm{inn}}),$$

The Kerr-Newman solution is a non-vacuum solution. It has two horizons, and it presents an ergosphere.

Kerr-Newman black holes

$$r_{\mathrm{h}}^{\mathrm{out}} = GMc^{-2} + [(GMc^{-2})^2 - a^2c^{-2} - q^2]^{1/2}.$$

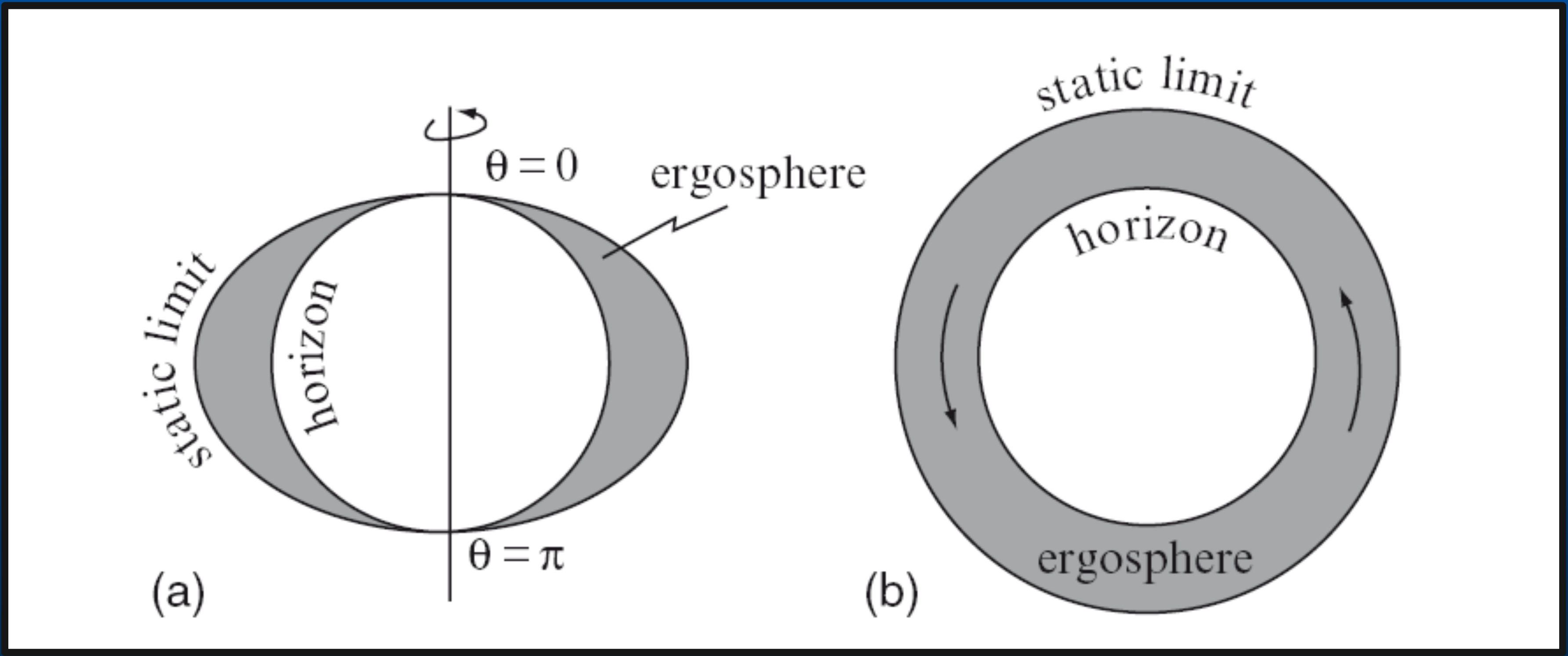
$$A = 4\pi(r_{\mathrm{h}}^{\mathrm{out}\,2} + a^2c^{-2}).$$

$$r_{\mathrm{h}}^{\mathrm{inn}} = GMc^{-2} - [(GMc^{-2})^2 - a^2c^{-2} - q^2]^{1/2}.$$

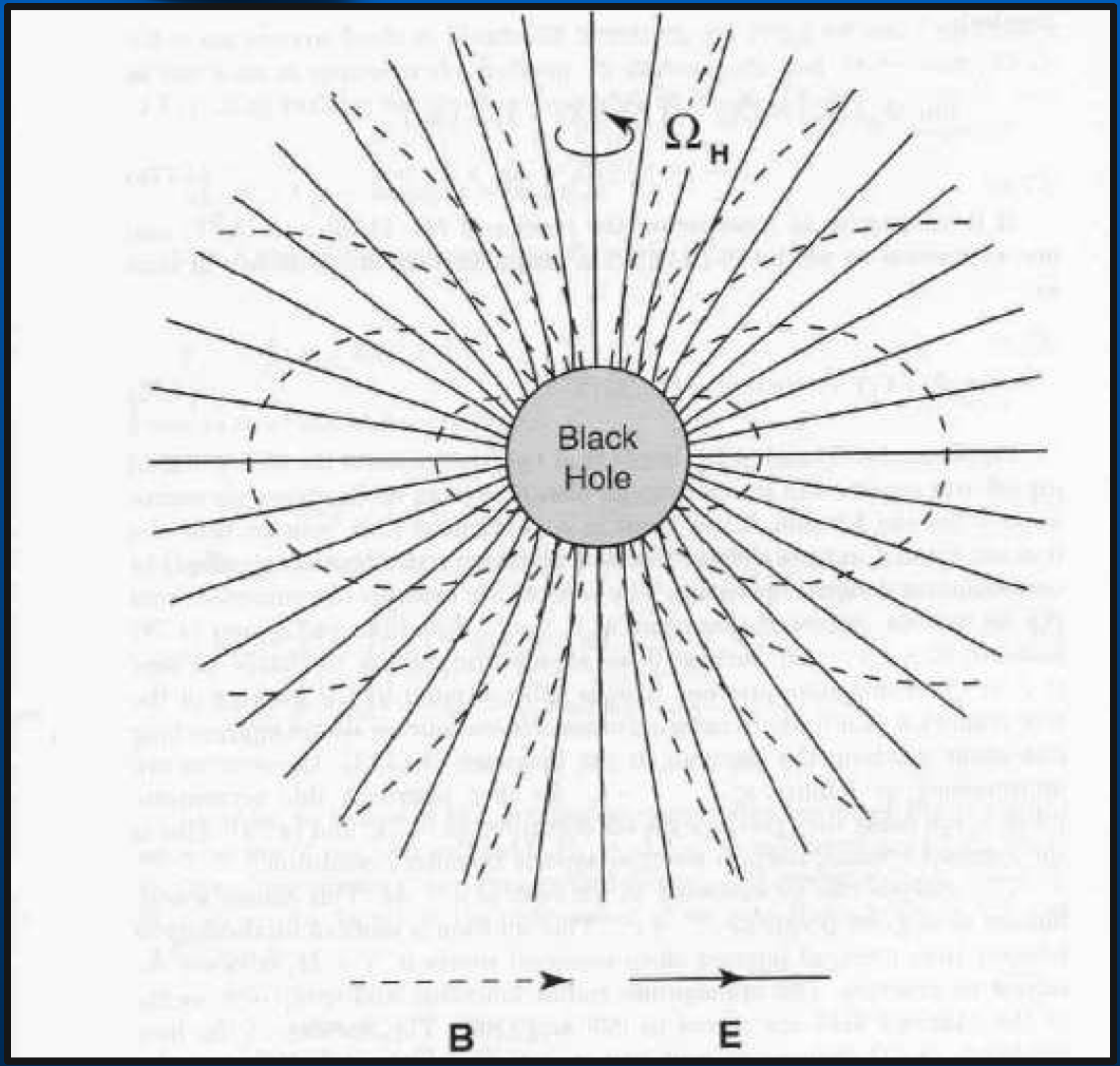
The Kerr-Newman metric represents the simplest stationary, axisymmetric, asymptotically flat solution of Einstein's equations in the presence of an electromagnetic field in four dimensions. It is sometimes referred to as an ``electro-vacuum'' solution of Einstein's equations. Any Kerr-Newman source has its rotation axis aligned with its magnetic axis.

Kerr-Newman black holes: ergosphere

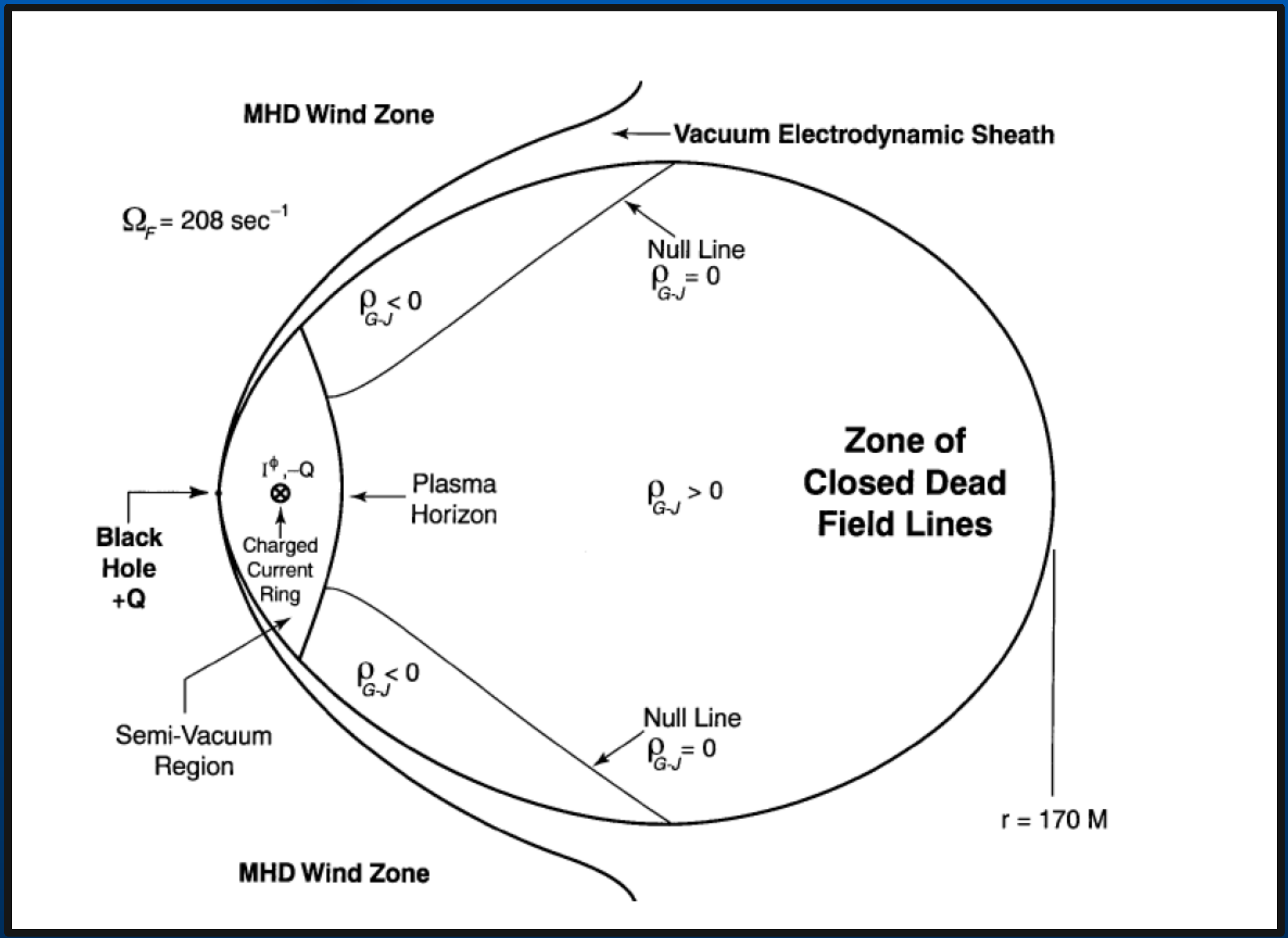
$$r_s = \frac{GM}{c^2} + \sqrt{\left(\frac{GM}{c^2}\right)^2 - a^2\cos^2\theta - e^2}$$



Kerr-Newman black holes



Kerr-Newman black holes



Fields around the ring singularity

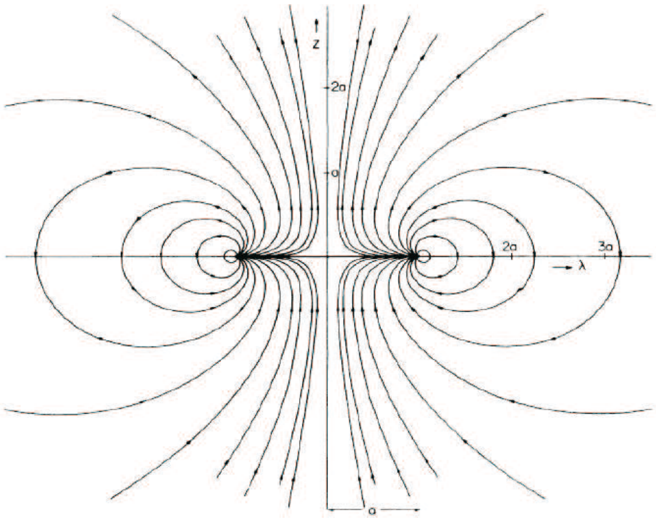


Figure 13. Magnetic field of a Kerr-Newman source. See text for units. From Pekeris & Frankowski (1987).

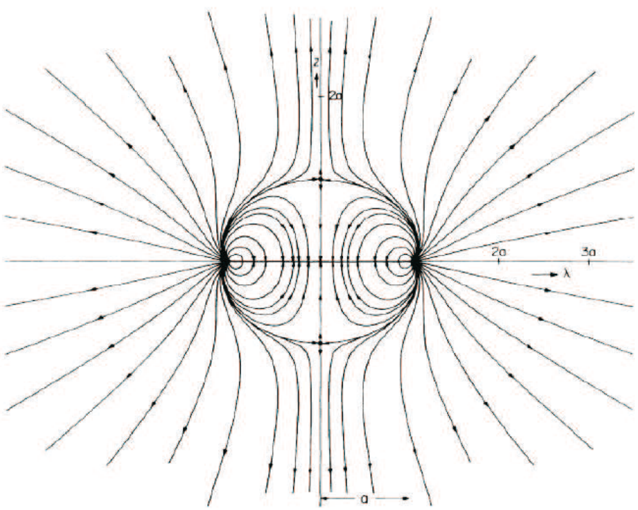
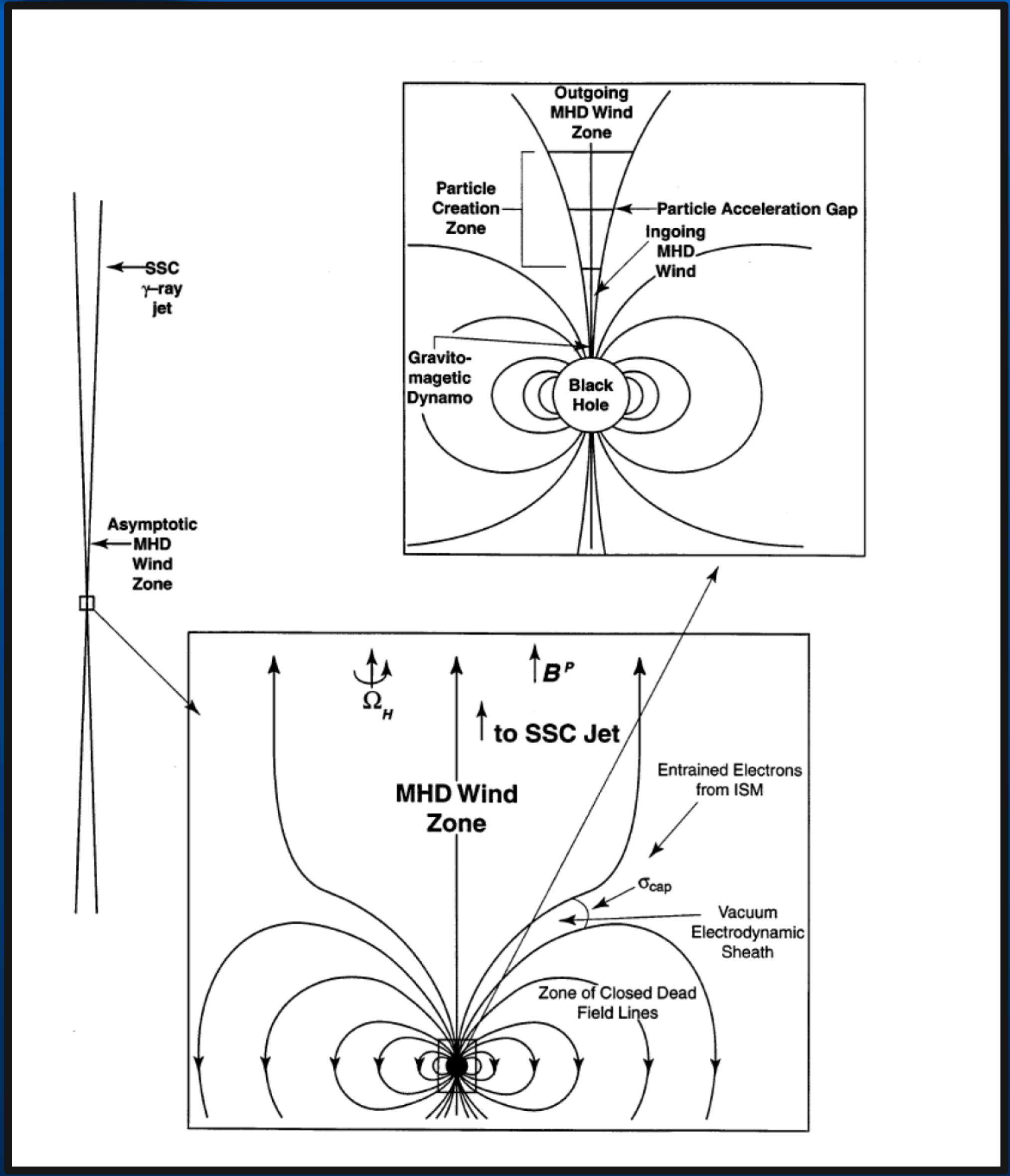


Figure 14. Electric field of a Kerr-Newman source. See text for units. From Pekeris & Frankowski (1987).

Kerr-Newman black holes



Punsly 1998
Punsly, Romero, et al. 2000

Regular black holes

- A regular black hole has no essential singularity.
- They can result if the collapse is stopped inside the event horizon.
- Possible causes: repulsive gravity, de Sitter interior, special equation of state, quantum gravity effects.

Regular black holes

$$p_r(\rho) = \left[\alpha - (\alpha+1)\left(\frac{\rho}{\rho_{\max}}\right)^m\right]\left(\frac{\rho}{\rho_{\max}}\right)^{1/n}\rho.$$

The maximum limiting density $\rho_{\max}$ is concentrated in a region of radius

$$r_0 = \sqrt{\frac{1}{G\rho_{\max}}}.$$

Regular black holes

At low densities $p_r \propto \rho^{1+1/n}$ and the equation reduces to that of a polytrope gas. At high densities close to ρ_{max} the equation becomes $p_r = -\rho$, and the system behaves as a gravitational field dominated by a cosmological term in the field equations.

$$p_r(\rho) = \left[\alpha - (\alpha+1)\left(\frac{\rho}{\rho_{max}}\right)^2\right]\left(\frac{\rho}{\rho_{max}}\right)\rho.$$

Regular black holes

$$ds^2 = -B(r)dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where

$$B(r) = \exp \int_{r_0}^{r} \frac{2}{r'^2}\left[m(r') + 4\pi r'^3 p_{sfr'}\right] \times \left[\frac{1}{(1 - \frac{2m(r')}{r'})}\right] dr',$$

and

$$m(r) = 4\pi \int_0^r \rho(r') r'^2 dr'.$$

Regular black holes

Outside the body $\rho \to 0$, and the metric becomes Schwarzschild solution for $R_{\mu\nu} = 0$. When $r \to 0$, $\rho = \rho_{\max}$ and the metric becomes of de Sitter type:

$$ds^2 = \left(1 - \frac{r^2}{r_0^2}\right)c^2 dt^2 - \left(1 - \frac{r^2}{r_0^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

with

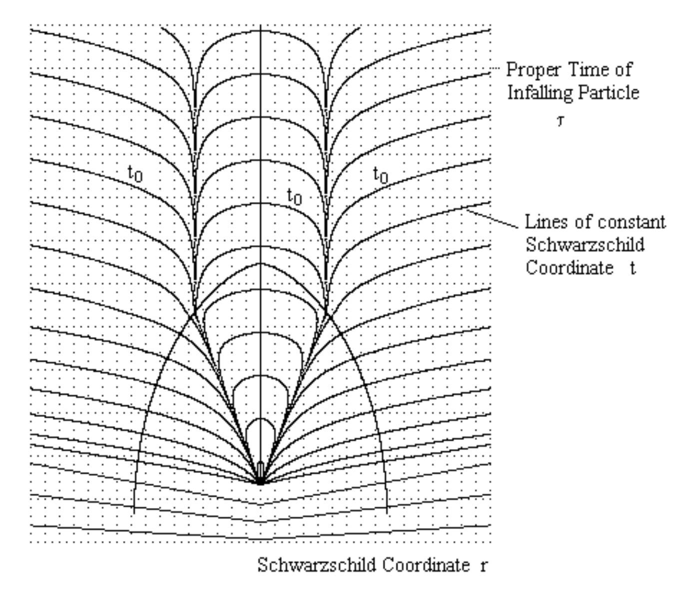
$$r_0 = \sqrt{\frac{3}{8\pi G \rho_{\max}}}.$$

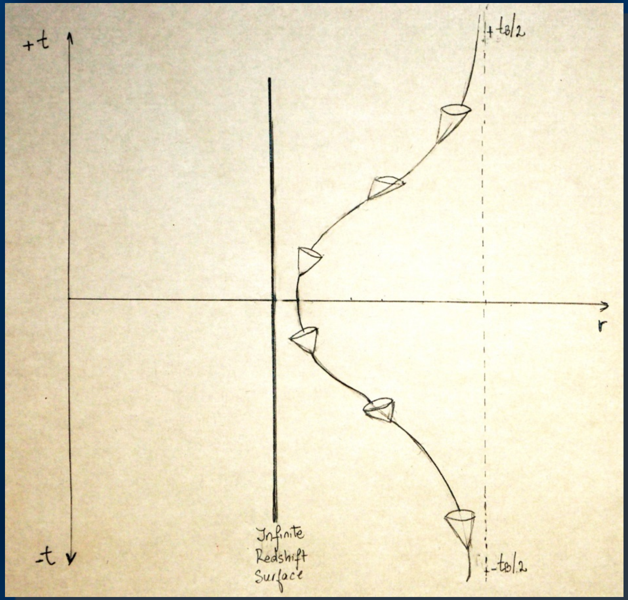
Regular black holes

There is no singularity at $r = 0$ and the black hole is regular. For $0 \leq r < 1$ it has constant positive density $\rho_{\max}$ and negative pressure $p_r = -\rho_{\max}$ and space-time becomes asymptotically de Sitter in the innermost region. It might be speculated that the transition in the equation of state occurs because at very high densities the matter field couples with a scalar field that provides the negative pressure.

Mimickers

Mimickers result if the collapse if stopped just outside the event horizon. Then a bounce can occur. If the bounce is sufficiently close to the horizon, the gravitational redshift would induce a time dilation such that for an observer in the infinite the collapse object would be not any more dynamical on timescales of the age of the universe, and for all practical purposes it would behave as a black hole.





Pérez & Romero (2016)